2021-08-31

BOUNDARY CONTROLLABILITY, STABILIZATION AND TRACKING PROBLEMS FOR PARABOLIC SYSTEMS

HERNÁNDEZ URRUTIA, ESTEBAN

UNIVERSIDAD TECNICA FEDERICO SANTA MARIA

Université Grenoble Alpes

THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ GRENOBLE ALPES

préparée dans le cadre d'une cotutelle entre la *Communauté Université Grenoble Alpes* et *l'Universidad Técnica Federico Santa María*

Spécialité : **Automatique-Productique**

Arrêté ministériel : 25 mai 2016

Présentée par

Esteban HERNÁNDEZ

Thèse dirigée par **Eduardo CERPA** et **Christophe PRIEUR**

préparée au sein du **Laboratoire Grenoble Image Parole Signal Automatique (Gipsa-Lab) et du Département de Mathématiques de l'UTFSM**

dans **l'École Doctorale Électronique Électrotechnique Automatique et Traitement du Signal (EEATS) et le Doctorat de l'UTFSM, spécialité Mathématique.**

Contrôlabilité frontière, stabilisation et poursuite pour des systèmes paraboliques

Boundary controllability, stabilization and tracking problems for parabolic systems

Thèse soutenue publiquement le **31 août 2021**,
devant le jury composé de :

Alberto MERCADO
Professeur Associé, Universidad Técnica Federico Santa María, Président

Lucie BAUDOUIN
Chargée de Recherche, CNRS, Laboratoire d'Analyse et d'Architecture de Systèmes, Examinatrice

Marius TUCSNAK
Professeur, Université de Bordeaux, Examinateur

Franck BOYER
Professeur, Université Toulouse III - Paul Sabatier, Rapporteur

Thomas MEURER
Professeur, Christian-Albrechts-Universität zu Kiel, Rapporteur

Christophe PRIEUR
Directeur de Recherche, CNRS, Université Grenoble Alpes, co-Directeur de thèse

Eduardo CERPA
Professeur Associé, Pontificia Universidad Católica de Chile, co-Directeur de thèse

"*There is a difference between doing some particular just or temperate action and being a just or temperate man. Someone who is not a good tennis player may now and then make a good shot. What you mean by a good player is the man whose eye and muscles and nerves have been so trained by making innumerable good shots that they can now be relied on. They have a certain tone or quality which is there even when he is not playing, just as a mathematician's mind has a certain habit and outlook which is there even when he is not doing mathematics. In the same way a man who perseveres in doing just actions gets in the end a certain quality of character.*"

C.S Lewis.

Acknowledgements

I would like to express my deepest gratitude to God, for his providence and cares with me.

I am deeply indebted to my thesis advisors. Eduardo Cerpa and Christophe Prieur. Thanks you for your patience, guidance, enthusiastic encouragement and useful critiques of this research work. Thanks for let me learn from you both.

I must also to thank to Patricio Guzmán, for his very good disposition to work and share your experience as researcher with me.

I would like to thank to the professors Franck Boyer and Thomas Meurer, to agree to write the reports for this thesis. Besides, I extend my gratitude to the professors, Lucie Baudouin, Alberto Mercado and Marius Tucsnak, for their good disposition to be part of the jury for the final dissertation.

Many thanks to the Departamento de Matemática, Universidad Técnica Federico Santa María and to the Gipsa-Lab, Université Grenoble Alpes, for provide me a very nice working space to do my research activities. I gratefully acknowledge the help of the administrative staff in both institutions, for their kindness every time which I needed help with the administrative paperwork related to the doctoral program. I cannot thank enough to my colleagues in both institutions for the great work environment.

This thesis could be develop thanks to the support of the fellowships *ANID, Beca Doctorado Nacional 21171188*, and *Programa de Incentivos a la Iniciación Científica, (PIIC)*, Universidad Federico Santa María, Valparaíso, Chile. Also, I gratefully acknowledge the support of FONDECYT 1180528, ANID Basal Project FB0008 and ANID Millennium Science Initiative Program NCN19-161. I extend my gratitude to Campus France Chile, for its support during my research stay at Gipsa-Lab in Grenoble.

I cannot begin to express my gratitude to my friends Pancho, Carlos, Francisco and Pamela. Thanks to you all your support. To share my joy and also hearing my frustrations. To my friends Diego, Sebastián, Louis and Lukas, thanks for welcoming me at your home, for your friendship and companionship.

During my research stay at Gipsa Lab in Grenoble, I had the opportunity to meet great people, The Gipsa Docs!, Thanks to you for your support and companionship during that time. A special mention to my friend and great mathematician Constantinos Kitsos.

I want to thank to the ICG, Pastor Blaine and his family, for their support during my stay at Grenoble and his constant cares during the first lockdown time in France. A special mention to my friend Jean Luc, thanks for all your book recommendations.

I extremely grateful to my family. My parents, Luis and Zenadia and my brother Lucas. For their constant support and prayers for me.

Résumé

La modélisation mathématique a un rôle clé dans la description d'une grande partie des phénomènes dans les sciences appliquées, les applications technologiques et industrielles.

Un modèle mathématique est un ensemble de relations mathématiques, généralement des équations, capables de décrire les caractéristiques essentielles d'un système naturel ou artificiel, dans le but de décrire, prévoir et contrôler son évolution.

Le but de cette thèse est d'étudier certains problèmes de contrôle dans des modèles mathématiques régis par des équations différentielles partielles de type parabolique. Dans le chapitre un on introduit une forme générale, les problèmes ètudiés et les résultats principause

Au chapitre deux, le modèle à particule unique est utilisé pour décrire le comportement d'une batterie Li-ion. L'objectif principal est de concevoir un courant d'entrée de rétroaction afin de réguler l'état de charge, denoté SOC par ses initiales en anglais, à une trajectoire de référence prescrite. Pour ce faire, nous utilisons la concentration ionique limite comme sortie. Tout d'abord, nous la mesurons directement puis nous supposons l'existence d'un estimateur approprié, qui a été établi dans la littérature à l'aide de mesures de tension. En appliquant la méthode de backstepping et les outils Lyapunov, nous sommes en mesure de construire des observateurs et de concevoir des contrôleurs de retour de sortie donnant une réponse positive au problème de suivi du SOC. Nous fournissons des preuves de convergence et effectuons des simulations numériques pour illustrer nos résultats théoriques.

Le chapitre trois est consacré à l'étude de la propriété de contrôlabilité frontière de certains systèmes paraboliques-elliptiques. Plus précisément, tout au long de ce chapitre, nous prouvons la propriété de contrôlabilité nulle pour deux systèmes paraboliques-elliptiques unidimensionnels. Les deux équations sont sous l'action d'un contrôle scalaire à la frontière. Dans un premier cas, nous étudions la contrôlabilité nulle pour un système avec un terme non linéaire dans la partie parabolique avec un contrôle placé à la frontière de l'équation parabolique. Dans un second cas, nous étudions un système linéaire avec le contrôle placé aux bords de l'équation elliptique. Les arguments, dans le premier cas, reposent sur le principe de dualité contrôlabilité-observabilité et une estimation de Carleman appropriée pour la solution de l'équation adjointe du système linéarisé. Ensuite, au moyen d'un théorème inverse local, nous prouvons le résultat pour le systeme non linéaire original. Pour le second cas, nous utilisons la méthode des moments et l'analyse spectrale de l'opérateur spatial sous-jacent associé à un tel système.

Au chapitre quatre, nous abordons le problème de la stabilisation rapide d'une équation de chaleur instable unidimensionnelle sous l'action d'une perturbation inconnue. En combinant la méthode de backstepping et l'opérateur à valeurs multiples sign$(\cdot)$, nous concevons une loi de rétroaction qui stabilise exponentiellement le système, dans la norme L^2. De plus, le taux de décroissance peut être fixé arbitrairement grand. L'existence de solutions du système en boucle fermée est obtenue en utilisant la théorie des opérateurs monotones maximaux et des simulations numériques sont effectuées afin d'illustrer nos résultats.

Enfin, au chapitre cinq, nous rassemblons quelques conclusions et remarques sur les chapitres précédents. En outre, nous discutons de certaines questions en ouvertes et de recherches futures, pour chacun de ces problèmes.

Mots-clés: Systèmes de contrôle, équations différentielles partielles, Problème de poursuite, méthode de Backstepping, contrôlabilité, estimations Carleman, Opérateurs Monotones Maximaux.

Abstract

Mathematical modeling has a key role in the description of a large part of phenomena in applied science, technological and industrial applications.

A mathematical model is a set of mathematical relations, usually equations, able to describe the essential features of a natural or artificial system, with the purpose to describe, forecast and control its evolution.

The goal of this thesis is to study some control problems in mathematical models governed by partial differential equations of parabolic type. In Chapter one we introduce in a general way the problems that have been studied and the obtained main results.

In Chapter two the Single Particle Model is used to describe the behavior of a Li-ion battery. The main goal is to design a feedback input current in order to regulate the State of Charge (SOC) to a prescribed reference trajectory. In order to do that, we use the boundary ion concentration as output. First, we measure it directly and then we assume the existence of an appropriate estimator, which has been established in the literature using voltage measurements. By applying the backstepping method and Lyapunov tools, we are able to build observers and to design output feedback controllers giving a positive answer to the SOC tracking problem. We provide convergence proofs and perform some numerical simulations to illustrate our theoretical results.

The Chapter three is devoted to study the boundary controllability property of some parabolic-elliptic systems. More precisely, along this chapter, we prove the null controllability property for two one-dimensional parabolic-elliptic systems. Both of them under the action of one scalar control at the boundary. In a first case, we study the null controllability for a system with a non-linear term in the parabolic part with a control placed at the boundary of the parabolic equation. In a second case, we study a linear system with the control placed at the boundary of the elliptic part. The arguments, in the first case, rely on the controllability-observability duality principle and a suitable Carleman estimate for the solution of the adjoint equation of the linearized system. Then, by means of a local inverse theorem we prove the result for the original system. For the second case, we use the moment method and the spectral analysis of the underlying spatial operator associated to such system.

In Chapter four, we address the problem of rapid stabilization of a one dimensional unstable heat equation under the action of an unknown boundary disturbance. Combining the backstepping method and the multivalued operator $sign(\cdot)$, we design a boundary feedback law which exponentially stabilizes the system, in the L^2 norm. Moreover, the rate can be fixed arbitrarily large. The existence of solutions to the closed-loop system is obtained by using the theory of maximal monotone operators and numerical simulations are performed in order to illustrate our results.

Finally, in Chapter five, we collect some conclusion and remarks on every problem studied in the previous chapters. Besides, we discuss some remaining open questions and future line research, for every one of those problems.

Keywords: Control systems, Partial Differential Equations, Output Tracking,

Backstepping Method, Controllability, Carleman estimate, Maximal Monotone Operators.

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Chapter 1

Introduction and main results

Summary

1.1 Introduction

Mathematical modelling has a key role in the description of large part of phenomena in applied science, technological and industrial applications.

A mathematical model is a set of mathematical relations, usually equations, able to describe the essential features of a natural or artificial system, with the purpose to describe, forecast and control its evolution. Partial differential equations (PDEs) have been successfully and widely used to derive some of these mathematical models. A control system governed by a PDE is a dynamic system in which it is possible to act by mean of suitable controls. There is a large variety of problems that arise when we study a control system. The most common problems are the controllability and the stabilization.

Controllability problem

Let H and U be two real Hilbert spaces, $T > 0$ and let us consider the following control system

$$\begin{cases} \dot{z} = Az + Bu, \quad t \in (0,T), \\ z(0) = z_0, \end{cases} \tag{1.1}$$

where the state is denoted by z and the control by u. Let assume that $A : D(A) \subset H \to H$ is a linear m-dissipative operator. Then by the Lumer-Phillips Theorem A is the infinitesimal generator of a strongly continuous semigroup of the linear operators $S_A(t)$. As usual, we denote the adjoint of A by A^*. Let assume that $u \in L^2(0,T;U)$ and $B \in \mathcal{L}(U; D(A^*)')$. Here, we have called as $\mathcal{L}(H_1; H_2)$ the space of linear continuous linear maps between H_1 and H_2 and the dual space of H as H'.

Under these assumptions and by a density argument, it is well known that for any $z_0 \in H$, there exists a unique $z \in C([0,T]; H)$ solution to (1.1). Moreover, the variation of constants formula holds.

$$z(t) = S_A(t)z_0 + \int_0^t S_A(t-s)Bu(s)\,\mathrm{d}s. \tag{1.2}$$

In this case, different notions of controllability can be formulated as follows :

- The system (1.1) is said to be **null controllable** at time T, if for any initial condition $z_0 \in H$, there exists a control $u \in L^2(0,T;U)$, such that the solution z to (1.1), satisfies that $z(T) = 0$.

- The system (1.1) is said to be **approximately controllable** at time T, if for any initial condition $z_0 \in H$, for any $\varepsilon > 0$, and for any final state $z_f \in H$ there exists a control $u \in L^2(0,T;U)$, such that the solution z to (1.1), satisfies that $\|z(T) - z_f\|_H \leq \varepsilon$.

- The system (1.1) is said to be **exactly controllable** at time T, if for any initial condition $z_0 \in H$ and any final state $z_f \in H$, there exists a control $u \in L^2(0,T;U)$, such that the solution z to (1.1), satisfies that $z(T) = z_f$.

At this point, it is worth to mention that, for the case of finite-dimensional linear control systems all these definitions are equivalent. On the other case, it is obvious that exact controllability implies null controllability and approximate controllability. When the system is linear, null controllability implies that the system is approximately controllable.

As we deal with parabolic equations, we will focus on the null controllability property of (1.1), which can be characterized by means of an observability inequality for the adjoint system. More precisely, the control system (1.1) is null controllable if and only if there there exists a positive constant C, such that

$$\|w(0)\|_H^2 \leq C \int_0^T \|B^* w\|_U^2 \,\mathrm{d}t, \tag{1.3}$$

for any $\bar{w} \in H$, and w solution to the following equation

$$\begin{cases} -\dot{w} - A^* w = 0, & t \in (0,T), \\ w(T) = \bar{w}. \end{cases} \tag{1.4}$$

Equation (1.4), is called the adjoint equation to the control system (1.1).

This is the controllability-observability duality principle, more details can be found in [17, 80].

Stabilization problem

Let us consider $u = 0$ in system (1.1). Let $\bar{z}$ be an equilibrium solution, that is $A\bar{z} = 0$, with $\bar{z} \in D(A)$. Then

- $\bar{z}$ is said to be **stable**, if for any $\varepsilon > 0$ there exist $\delta > 0$, such that the solution z to (1.1), satisfies that for any initial z_0 in the ball $B(\bar{z}, \delta)$, it holds that $z(t) \in B(\bar{z}, \varepsilon)$, for all $t \geq 0$.

- $\bar{z}$ is said be **asymptotically stable**, if $\bar{z}$ is stable and there exists $\delta > 0$, such and for any $z_0 \in B(\bar{z}, \delta)$, the solution $z(t)$ to (1.1), satisfies $z(t) \to \bar{z}$, as $t \to \infty$.

If $\bar{z}$ is not stable, we said that $\bar{z}$ is unstable.

Now, if $\bar{z}$ is asymptotically stable and it holds that, for all $z_0 \in H$, the solution to (1.1) satisfies

$$\|z(t) - \bar{z}\|_H \leq e^{-\mu t}\|z(0) - \bar{z}\|_H, \quad \forall t \geq 0. \tag{1.5}$$

we said that $\bar{z}$ is **exponentially stable**, with decay rate $\mu > 0$.

The stabilization problem, can be formulated as follows. Let z_{ref} be a desired state, called reference. How to design a control u such that the pair (z_{ref}, u_{ref}) which satisfies $0 = Az_{ref} + Bu_{ref}$ be stable, or asymptotically stable or exponentially stable?

Output tracking problem

Let us consider the following control system.

$$\begin{cases} \dot{z} = Az + Bu, & t \in (0, \infty), \\ y = Cz, & t \in (0, \infty), \\ z(0) = z_0, \end{cases} \tag{1.6}$$

where the state of the system is denoted by z, u is the control, C is a bounded or unbounded operator and y is the output of the system. The output tracking problem can be formulated as follows. Given a reference for the output y, namely y_{ref}, we look for controls such that $y(t) \to y_{ref}$, as $t \to \infty$, in a suitable norm. Moreover, in this thesis, we are interested in looking for controls in the feedback form. That is, $u = Kz$, with a full state measurement of z or with a partial measure of the state z.

Stabilization under disturbances

Usually when the stabilization problem is studied for a control system, this is assumed under idealized conditions. However, it is not always is possible to ensure those conditions. That leads us to study control systems under disturbances, that is, to take into consideration external sources of instability. Consider, for instance, the following control system

$$\begin{cases} \dot{z} = Az + Bu + d(t), & t \in (0, \infty), \\ z(0) = z_0, \end{cases} \tag{1.7}$$

where the state is denoted by z, u is the control and d is an unknown disturbance. Let z_{ref} be a desired state, called reference. The main task here is to design a control u in order to reject or attenuate the effects of the disturbance d on the system, at the same time to force the system to converges to the reference z_{ref}.

In this thesis, we address these control issues for some mathematical models given by parabolic PDEs. In the remain part of this introductory chapter, we summarize the most important results obtained for each one of those control problems. More precisely, in Section 1.2, is introduced an output tracking problem in the context of an electrochemical ion battery modeled by a parabolic PDE. The main results presented at that section are fully developed and proved along Chapter 2. Then, in Section 1.3, the problem of boundary null controllability for a parabolic-elliptic system is presented. Besides, the main contributions are summarized in two theorems. The null controllability problem is studied in Chapter 3. Finally, in Section 1.4, the problem to stabilize a general unstable heat equation under boundary disturbance is stated. The feedback law and the main result about the stability for the system in closed loop are presented, as well.

1.2 A tracking problem in a battery model

Today batteries are being developed to power a crescent and wide range of applications as laptops, smartphones, watches, electric vehicles, medicals devices and many others. Consequently, batteries are certainly in the middle of the technological development [3]. In this direction, li-ion batteries are gaining more and more attention due to its very good properties, compared to alternative battery technologies. For example, li-ion batteries provide one of the best energy-weight ratios and have a low self discharge when not in use [12].

An intelligent battery control system can ensure longevity and performance of battery, but such a type of improvements relies in an exhaustive understanding of energy storage. Thus, the modeling of li-ion batteries has a key role in the design of battery management systems.

Along chapter 2, is studied the problem of tracking of the State of Charge in a battery modeled by an electrochemical model. In other words, given a reference State of Charge profile, we want to find the appropriate input current in order to get the real State of Charge near to the given reference.

The model used to describe the battery behavior is called the Single Particle Model (SPM). The SPM considers each electrode as a single spherical particle and neglects the electrolyte dynamics. Let $c(t,r)$ be the ion concentration in the negative electrode. Then, its evolution is modeled by the following diffusion equation

$$\begin{cases} c_t = \frac{2}{r}c_r(t,r) + c_{rr}(t,r), & (t,r) \in \bar{Q}, \\ c_r(t,0) = 0, \quad c_r(t,1) = \tilde{\rho}I(t), & t \in (0,\infty), \\ c(0,r) = c_0(r), & r \in (0,1), \end{cases} \tag{1.8}$$

where $\bar{Q} = (0,\infty) \times (0,1)$, $\tilde{\rho}$ is a group of electrochemical parameters of the model, see section 2.1 for more details. $c_0(r)$ is the initial condition, and $I(t)$ is the input current, which is used as a control.

The State of Charge is defined by

$$SOC(t) = \frac{3}{c_{\max}} \int_0^1 c(t,r) r^2 \, \mathrm{d}r. \tag{1.9}$$

In this work, we deal with the regulation problem for the State of Charge. To do that, we aim to apply the main idea of the certainty equivalence principle or separation principle, which refers to the fact that plug-in a convergent estimator in a stable closed-loop system does not change the stability. At this point, it is important to mention that, for the case of infinite-dimensional systems, the certainty equivalence principle may not apply as we know for the case of finite-dimensional systems.

The control design, developed along Chapter 2, can be summarized as follows. We begin by dealing with the most simple case. We assume that we are able to measure the full state $c(t,r)$ of (1.8), then we design an output feedback control which achieves the tracking. After that, using the backstepping method, see for example [49, 82], we design an output feedback depending on the measure of the boundary concentration $c(t,1)$. The next step in our design consists of replacing the boundary measure $c(t,1)$ by a convergent observer, namely $\varphi(t)$.

As far as we know, in the literature this separation principle is used without proof. In [66] the authors propose an adaptive scheme to obtain $\varphi(t)$ based on the continuous Newton method. The proof of convergence of the scheme and of the closed-loop system is omitted. The authors of [64] state an exponential convergent scheme to obtain φ, but the proof of convergence of the system in closed loop is omitted.

In order to state the separation principle, we assume that this estimator $\varphi(t)$ satisfies the following assumption.

Assumption 1.2.1. *There exist a function $\varphi : [0,\infty) \to \mathbb{R}$ and positive constants L and μ such that*

$$|\varphi(t) - c(t,1)| \le Le^{-\mu t}, \quad \forall t \ge 0,$$

where $c(t,r)$ is the solution to (1.8).

Let us define, for some $p_1(r,\lambda)$ and $p_0(\lambda)$ (given later by the backstepping method), the following copy of the plant

$$\begin{cases} \partial_t \widehat{c_\varphi} = \frac{2}{r}\partial_r \widehat{c_\varphi} + \partial_{rr}\widehat{c_\varphi} + p_1(r,\lambda)(\varphi(t) - \widehat{c_\varphi}(t,1)), \\ \partial_r \widehat{c_\varphi}(t,0) = 0, \quad \partial_r \widehat{c_\varphi}(t,1) = \tilde{\rho} I(t) + p_0(\lambda)(\varphi(t) - \widehat{c_\varphi}(t,1)), \\ \widehat{c_\varphi}(0,r) = \widehat{c_\varphi}_0(r). \end{cases} \tag{1.10}$$

Our first result consists in the exponential stability of the observer error $\widetilde{c}(t,r) = c(t,r) - \widehat{c_\varphi}(t,r)$, which is stated in Theorem 1.2.2. This constitutes the main contribution of Chapter 2, which presents rigorous proofs of our statements on convergence.

Theorem 1.2.2. *Consider $\varphi : [0,\infty) \to \mathbb{R}$ and constants $L > 0$ and $\mu > 0$ satisfying Assumption 1.2.1, the initial condition $\widetilde{c}_0 = c_0(r) - \widehat{c_\varphi}_0(r)$ and the gains $p_0(\lambda)$ and $p_1(r,\lambda)$ given by*

$$p_0(\lambda) = \frac{\lambda}{2} \tag{1.11}$$

and

$$p_1(r,\lambda) = \left(\frac{\lambda}{(r^2-1)} + \frac{\lambda}{2}\right) J_2\left(\sqrt{\lambda(r^2-1)}\right) - \frac{\lambda}{2} J_0\left(\sqrt{\lambda(r^2-1)}\right), \tag{1.12}$$

where J_0 and J_2 are the zero and second order Bessel functions of first kind respectively.

Therefore there exists $\lambda_{\sup} > 2+\sqrt{6}$ such that for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$ the function $\tau(\lambda)$ defined by

$$\tau(\lambda) = \frac{\pi^2}{2} - \frac{2}{\lambda}\|p_1(\cdot,\lambda)\|^2_{L^2_r(0,1)} \tag{1.13}$$

is positive. Moreover, depending on μ, the L^2_r norm of the observer error $\widetilde{c}(t,r)$ satisfies one of the following cases:

1. *If $\mu > \frac{\tau(\lambda)}{2}$ for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$, then*

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)} \le \left(2\|\widetilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda) - 2\mu|}\right) e^{-\tau(\lambda)t}, \ \forall t \ge 0, \forall \lambda \in [2+\sqrt{6}, \lambda_{\sup}). \tag{1.14}$$

2. *If* $\mu = \frac{\tau(\bar\lambda)}{2}$, *for some* $\bar\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$, *then*

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\bar\lambda^3+4\bar\lambda)}{2}t\right)e^{-\tau(\bar\lambda)t}, \quad \forall t \geq 0. \tag{1.15}$$

The followings results are a direct consequence of Theorem 1.2.2 and describe the performance of observer $\widehat{c}_\varphi(t,r)$.

Corollary 1.2.3. *Let* $\lambda^* = 2+\sqrt{6}$. *Depending on* μ *we have the following*

1. *if* $2\mu > \tau(\lambda^*)$, *then the highest decay rate of* $\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)}$ *is* $\tau(\lambda^*)$ *and the transient state is bounded. Moreover, it holds*

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)} \leq 2\|\tilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\lambda^{*3}+4\lambda^*)}{2|\tau(\lambda^*)-2\mu|}, \quad \forall t \geq 0, \tag{1.16}$$

2. *if* $2\mu \leq \tau(\lambda^*)$, *then the decay ratio of* $\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)}$ *is* 2μ *and the transient state is bounded. Moreover, it holds*

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)} \leq \frac{L^2(\bar\lambda^3+4\bar\lambda)}{2\tau(\bar\lambda)}\exp\left\{\frac{4\|\widetilde{c}_0\|^2_{L^2_r(0,1)}\tau(\bar\lambda)}{L^2(\bar\lambda^3+4\bar\lambda)} - 1\right\}, \quad \forall t \geq 0, \tag{1.17}$$

where $\bar\lambda$ *is solution to equation* $2\mu = \tau(\bar\lambda)$.

Let us define the following

$$N_1(\lambda) = 2\|\tilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}.$$

Corollary 1.2.4. *Let* $\lambda^* = 2+\sqrt{6}$ *and* $[\lambda^*, \lambda_{\text{sup}}]$ *the interval given by Theorem 1.2.2. If* $2\mu > \tau(\lambda^*)$, *then*

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)} \leq 2\|\tilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\bar\lambda^3+4\bar\lambda)}{2|\tau(\bar\lambda)-2\mu|}, \quad \forall t \geq 0, \tag{1.18}$$

where $\bar\lambda = \underset{\lambda\in[\lambda^*,\lambda_{\text{sup}}]}{\arg\min}\ N_1(\lambda)$ *and the decay ratio is given by* $\tau(\bar\lambda)$.

From the previous exponential stability result for $\widetilde{c}$ stated in Theorem 1.2.2, we are able to prove the following result.

Theorem 1.2.5. *Consider* $\varphi : [0,\infty) \to \mathbb{R}$ *and constants* $L > 0$ *and* $\mu > 0$ *satisfying Assumption 1.2.1, gains* $p_0(\lambda)$ *and* $p_1(r,\lambda)$ *given by* (1.11) *and* (1.12) *respectively. There exists* $\lambda_{\text{sup}} > 2+\sqrt{6}$ *such that for* $\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$ *we define the input current*

$$I(t) = \frac{c_{\max}}{3\tilde\rho}\left(\dot{SOC}_{ref}(t) + \gamma\left(SOC_{ref}(t) - \widehat{SOC}_\varphi(t)\right)\right), \tag{1.19}$$

where

$$\widehat{SOC}_\varphi(t) = \frac{3}{c_{\max}}\int_0^1 \widehat{c_\varphi}(t,r)r^2\,dr,$$

$\gamma > 0$ *is a design parameter and* $\widehat{c}_\varphi(t,r)$ *is the solution to* (1.10). *This feedback control* $I(t)$ *forces the system to satisfy*

$$|SOC_{ref}(t) - SOC(t)| \to 0, \quad t \to \infty \tag{1.20}$$

with an exponential rate, depending on the parameters.

The Chapter 2 is devoted to the proof of this statements. Besides, it can be found, in Section 2.5, some numerical simulations in order to illustrate the theoretical results.

1.3 Null controllability of some parabolic-elliptic systems

In Chapter 3 is studied the boundary null controllability of two kind of parabolic-elliptic systems.

In a first case, we consider the following control system

$$\begin{cases} z_t - z_{xx} + qz = f(z) + \zeta, & (t,x) \in (0,T) \times (0,L), \\ -\zeta_{xx} + \gamma\zeta = z & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = u(t), \quad z(t,L) = 0, & t \in (0,T), \\ \zeta(t,0) = 0, \quad \zeta(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{1.21}$$

where $T > 0$, $L > 0$, the state is given by (z,ζ), γ, $q \in L^\infty(0,L)$, $f \in W^{2,\infty}(\mathbb{R})$ is a nonlinear function and the time-dependent function u is a boundary control acting on the parabolic boundary condition.

In a second case, we consider the system given by

$$\begin{cases} z_t - z_{xx} + q_0 z = \zeta, & (t,x) \in (0,T) \times (0,L), \\ -\zeta_{xx} + \gamma_0\zeta = z & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = 0, \quad z(t,L) = 0, & t \in (0,T), \\ \zeta(t,0) = u(t), \quad \zeta(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{1.22}$$

where $T > 0$, $L > 0$, the state is given by (z,ζ), γ_0, q_0 are scalar constants, and the time-dependent function u is a boundary control acting on the boundary of the elliptic equation.

For the systems (1.21) and (1.22), we are interested in studying the null controllability by the action of a one single control placed at the boundary. That is, given $T > 0$ and appropriate space X, we say that system (1.21) or (1.22) is null controllable if for any initial condition $z_0 \in X$, there exists a boundary control u such that the solution to (1.21) or (1.22) with $z(0,\cdot) = z_0$ satisfies $(z(T,\cdot), \zeta(T,\cdot)) = (0,0)$.

To study the systems (1.21) and (1.22) let us introduce the following operator

$$F_\gamma : g \in L^2(0,L) \longmapsto F_\gamma(g) = \zeta \in H_0^1(0,L), \tag{1.23}$$

where ζ is the solution to

$$\begin{cases} -\zeta_{xx} + \gamma\zeta = g, \quad x \in (0,L), \\ \zeta(0) = 0, \zeta(L) = 0, \end{cases} \tag{1.24}$$

with $\gamma \in L^\infty(0,L)$. The operator F_γ is well defined, linear continuous, with continuity constant $C(\gamma_0, L)$, and self-adjoint. In consequence, it is possible to re-write system

(1.21) in a equivalent way, as follows

$$\begin{cases} z_t - z_{xx} + q(x)z = f(z) + F_\gamma(z), & (t,x) \in (0,T)\times(0,L),\\ z(t,0) = u(t), \quad z(t,L) = 0, & t \in (0,T),\\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{1.25}$$

To prove the null controllability of the system (1.25), as usual in this kind of problems, we begin by proving the boundary null controllability of the following linear system

$$\begin{cases} z_t - z_{xx} + (q - f'(0))z - F_\gamma(z) = g(t,x), & (t,x) \in (0,T)\times(0,L),\\ z(t,0) = u(t), \quad z(L,t) = 0, & t \in (0,T),\\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{1.26}$$

where F_γ is the operator given by (1.23), and g is an external force. Notice that if $g = 0$ we recover the linearized system of (1.25) around $z = 0$.

In order to prove the boundary null controllability of the linear system (1.26), we use the controllability-observability duality principle. To do that, we use the Carleman estimate with boundary observation to deduce an observability inequality for the adjoint system to (1.26), see Chapter 3, Section 3.3.1. Then, we show that the local boundary null controllability property holds for the nonlinear control system (1.25) by using a local inverse function argument. The first main result of Chapter 3 can be summarized as follows.

Theorem 1.3.1. *Let $T > 0$, $L > 0$, $\gamma, q \in L^\infty(0,L)$, $f \in W^{2,\infty}(\mathbb{R})$ such that $\gamma(x) \geq \gamma_0 > -\pi^2/L^2$, for all $x \in [0,L]$ and $q(x) \geq q_0$ such that $q_0 + f'(0) \geq C(\gamma_0, L)L/\pi - (\pi/L)^2$, for all $x \in [0,L]$. Then, the system (1.25) is locally null controllable. That is, there exists $r > 0$ such that for any $z_0 \in H^{-1}(0,L)$ such that $\|z_0\|_{H^{-1}(0,L)} \leq r$, there exists $u \in L^2(0,T)$ and $z \in C([0,T]; H^{-1}(0,L)) \cap L^2(0,T; L^2(0,L))$ solution to (1.25). Moreover it holds $z(T,x) = 0$.*

Now, in order to study the boundary null controllability for the system (1.22) we introduce a lift function $\xi \in C^2([0,L])$, such that $\xi(0) = 1$ and $\xi(L) = 0$ and let define the following change of variable

$$\tilde{\zeta}(t,x) = \zeta(t,x) - \xi(x)u(t). \tag{1.27}$$

Then, by using the operator F_{γ_0}, the system (1.22) can be rewritten as follows

$$\begin{cases} z_t - z_{xx} + q_0 z - F_{\gamma_0}(z) = \theta u, & (t,x) \in (0,T)\times(0,L),\\ z(t,0) = 0, \quad z(t,L) = 0, & t \in (0,T),\\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{1.28}$$

where $\theta = \xi - F_{\gamma_0}(-\xi_{xx} + \gamma_0 \xi)$. Now, we summarize the second main result of Chapter 3.

Theorem 1.3.2. *Let $T > 0$, $L > 0$, and constant coefficients q_0 and γ_0 such that $\gamma_0 > -\pi^2/L^2$, and q_0 such that $q_0 \geq C(\gamma_0, L)L/\pi - (\pi/L)^2$. Then system (1.28) is null controllable. That is, for any $z_0 \in L^2(0,L)$ there exist $u \in L^2(0,T)$ and $z \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^1(0,L))$ solution to (1.28). Moreover it holds $z(T,\cdot) = 0$.*

The moment method has been used to prove Theorem 1.3.2, see Chapter 3, Section 3.3.3.

1.4 Stabilization of a heat equation under disturbance

Let $L \in (0,\infty)$ and $a \in C^1([0,L])$. Let us consider

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty) \times (0,L), \\ z_x(t,0) = 0, & t \in (0,\infty), \\ z_x(t,L) = u(t) + d(t), & t \in (0,\infty), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{1.29}$$

In (1.29) the state of the system is denoted by $z = z(t,x)$, the boundary feedback law by $u(t)$ and the unknown boundary disturbance by $d(t)$.

As far as the undisturbed case is concerned, which is when the disturbance is zero, the sources of instability of (1.29) are its boundary conditions and $a^+(x) = \max\{a(x), 0\}$ (the non-negative part of a). In that case the rapid stabilization problem for (1.29) has been successfully solved in [55] with a boundary feedback law designed by means of the backstepping method and Lyapunov techniques. Such a feedback law is given by [55, equation (3.3)] and reads as

$$u(t) = -k(L,L)z(t,L) - \int_0^L k_x(L,s)z(t,s)\,\mathrm{d}s, \tag{1.30}$$

where $k = k(x,s)$ is a C^2 function on the triangle $\Omega = \left\{(x,s) \in \mathbb{R}^2 \ / \ 0 \le s \le x \le L\right\}$ being the unique solution to

$$\begin{cases} k_{xx}(x,s) - k_{ss}(x,s) = (a(s) + \omega)k(x,s), & (x,s) \in \Omega, \\ k_s(x,0) = 0, & x \in [0,L], \\ k(x,x) = \frac{1}{2}\int_0^x (a(s)+\omega)\,\mathrm{d}s, & x \in [0,L], \end{cases} \tag{1.31}$$

where $\omega > 0$ is a constant which fixes the rate for the exponential decay of the target system.

However, in the disturbed case it is uncertain whether we can employ (1.30), since in the construction of the gain kernel, based on the application of the method of successive approximations, see for instance [49, Chapter 4], to solve (1.31), no information of the disturbance is used, and hence, (1.30) might not be able to handle the effects of it. Accordingly, in (1.29) we may regard the disturbance as another source of instability and a new boundary feedback law is required to solve the problem under consideration.

The feedback design proposed, cancels the effects of the disturbances by using, in a suitable way, the multivalued operator $\mathrm{sign}(\cdot)$, defined by

$$\mathrm{sign}(f) = \begin{cases} \frac{f}{|f|} & \text{if} \quad f \neq 0, \\ [-1,1] & \text{if} \quad f = 0. \end{cases} \tag{1.32}$$

Even if in our analysis we consider an unknown boundary disturbance, we still need to establish some basic assumptions, which are:

(A1) There exists $D \in (0,\infty)$ such that

$$|d(t)| \le D, \quad \forall t \in [0,\infty) \tag{1.33}$$

(A2) The disturbance d satisfies the following regularity assumption

$$d \in W^{2,1}(0,\infty) \text{ and } d(0) = 0. \tag{1.34}$$

The Assumption **(A1)** is required for the design of a boundary feedback law able to handle the effects of an unknown boundary disturbance while **(A2)** is required for the proof of the well-posedness of the corresponding closed-loop system.

We are interested into the rapid stabilization problem for an unstable heat equation with an unknown boundary disturbance. In other words, a boundary feedback law is designed so that the corresponding closed-loop system is exponentially stable in $L^2(0,L)$, with decay rate as large as desired. The main result of the Chapter 3 is summarized as follows.

Theorem 1.4.1. *Let $a \in C^1([0,L])$, $\omega > 0$. Let us assume **(A1)** and **(A2)**. Let $k = k(x,s)$ be the gain kernel obtained from (1.31). For a regular enough function $f = f(t,x)$ let us introduce the boundary feedback law*

$$\begin{aligned} u(t,f) = -k(L,L)f(t,L) - \int_0^L k_x(L,s)f(t,s)\,ds \\ - D sign\left(f(t,L) + \int_0^L k(L,s)f(t,s)\,ds \right). \end{aligned} \tag{1.35}$$

Let us take an initial condition z_0 in the following set

$$\left\{ z_0 \in H^2(0,L), \text{ such that } y_0'(0) = 0 \text{ and } y_0'(L) + D sign(y_0(L)) \ni 0 \right\}, \tag{1.36}$$

where we have introduced

$$y_0(x) = z_0(x) + \int_0^x k(x,s) z_0(s)\,ds. \tag{1.37}$$

Then, there exists a unique $z = z(t,x)$ in $W^{1,1}(0,\infty; L^2(0,L)) \cap L^1(0,\infty; H^2(0,L))$ such that

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty) \times (0,L), \\ z_x(t,0) = 0, & t \in [0,\infty), \\ z_x(t,L) \ni u(t,z) + d(t), & t \in [0,\infty), \\ z(0,x) = z_0(x), & x \in [0,L]. \end{cases} \tag{1.38}$$

Moreover, (1.38) is exponentially stable in $L^2(0,L)$, with decay rate given by ω. In other words, given $\omega > 0$ the solution z to the closed-loop system (1.38) satisfies that

$$\|z(t,\cdot)\|_{L^2(0,L)} \leq C e^{-\omega t} \|z_0\|_{L^2(0,L)}, \quad \forall t \in [0,\infty). \tag{1.39}$$

The Chapter 4 of this thesis is dedicated to the feedback design and to prove Theorem 1.4.1. Besides, we perform numerical simulations in order to illustrate our theoretical results.

Finally, we want to point out certain connections between the individual topics. For instance, the Backstepping method is used to solve the control problem in Chapter 2 and Chapter 3. Another important tool, which are energy estimations for parabolic equation, were used in different contexts, for instance, in Chapter 2 to state the stability of the error observer, while in Chapter 3, were used to obtain well-posedness results for the control systems. Moreover, a Carleman estimate, which is a kind of energy weighted estimation, was used to get the observability inequality. In Chapter 4, an energy estimation is used to design a disturbance compensator to reject the disturbance signal.

1.5 Organization

The remaining chapters of this thesis are organized as follows. In Chapter 2, we prove the results presented in Section 1.2, related to the tracking problem for the state of charge in a li-ion battery model. Chapter 3 is devoted to the problem of null controllability of some parabolic elliptic systems under the action of one single control, see Section 1.3. The last problem addressed, presented in Section 1.4, is developed in Chapter 4. Finally in Chapter 5 we collect some remarks and conclusions for all these problems.

Regarding with the works that composes this thesis, we indicate that Chapter 2 is content in

- E. Hernández, C. Prieur, and E. Cerpa. A tracking problem for the state of charge in an electrochemical li-ion battery model. *Mathematical Control and Related Fields*, 2021. To appear.

The Chapter 3 is content in the following submitted article

- E. Hernández, C. Prieur, and E. Cerpa. Boundary null controllability of some parabolic-elliptic systems. *Submitted*, 2021.

The results in Chapter 4 were obtained in collaboration with Patricio Guzmán [1] and will be submitted for publication.

1.6 Notation

Let $C([0,L])$ be the set of the continuous functions $f:[0,L]\to\mathbb{R}$. Endowed with the norm

$$\|f\|_{L^\infty(0,L)} = \max_{x\in[0,L]} |f(x)|, \tag{1.40}$$

for which $C([0,L])$ is a Banach space.

Now, let k be a positive integer, we consider $C^k([0,L])$ as the set of the differentiable functions up to order k $f:[0,L]\to\mathbb{R}$. Endowed with the norm,

$$\|f\|_{C^k([0,L])} = \|f\|_{L^\infty(0,L)} + \sum_{j=1}^{k} \|f^{(j)}\|_{L^\infty(0,L)}, \tag{1.41}$$

the set $C^k([0,L])$ is a Banach spaces.

We introduce the following notation

$$\|f\|_{L^2_r(0,L)} = \left(\int_0^L f^2(r) r^2 \,\mathrm{d}r\right)^{1/2}, \tag{1.42}$$

$$\|f\|_{H^1_r(0,L)} = \|f\|_{L^2_r(0,L)} + \|f'\|_{L^2_r(0,L)}, \tag{1.43}$$

[1] email: *patricio.guzmanm@usm.cl*, Departamento de Matemática, Universidad Federico Santa María, Valparaiso, Chile.

$$\|f\|_{H_r^2(0,L)} = \|f\|_{L_r^2(0,L)} + \|f'\|_{L_r^2(0,L)} + \|f''\|_{L_r^2(0,L)}, \tag{1.44}$$

in order to consider the weighted spaces, with weight function r^2, denoted by $L_r^2(0,L)$, $H_r^1(0,L)$ and $H_r^2(0,L)$ and its norms respectively. When the weight function is identically equal to one, we omit the r index.

Let $C_c^\infty([0,L])$ be the space of infinitely differentiable functions with compact support on $[0,L]$. Then, $H_0^1(0,L) = \overline{C_c^\infty([0,L])}^{\|\cdot\|_{H^1(0,L)}}$. In virtue of the Poincaré inequality, we have consider the corresponding norm as

$$\|f\|_{H_0^1(0,L)} = \|f'\|_{L^2(0,L)}.$$

Besides, we have consider that $\|f\|_{L^2(0,L)} + \|f''\|_{L^2(0,L)}$ is a norm equivalent to the norm $\|f\|_{H^2(0,L)}$, see [7, Chapter 8] for instance.

Let X and Y be two normed vectorial spaces. Then, the norm of the space $X \cap Y$ is defined as $\|f\|_{X\cap Y} = \|f\|_X + \|f\|_Y$, for all $f \in X \cap Y$.

Let $T \in [0,\infty)$ and H be a real Hilbert space. Consider the function

$$\begin{aligned} z : [0,T] &\to H \\ t &\mapsto z(t,\cdot). \end{aligned} \tag{1.45}$$

Then, $C([0,T];H)$, denotes the space of continuous functions $z : [0,T] \to H$. Endowed with the norm

$$\|z\|_{L^\infty(0,T;H)} = \max_{t\in[0,T]} \|z(t,\cdot)\|_H, \tag{1.46}$$

for which $C([0,T];H)$ is a Banach space.

In analogous way, for $1 \le p < \infty$, we define the space $L^p(0,T;H)$ as the set of functions such that

$$\|z\|_{L^p(0,T;H)} = \left(\int_0^T \|z(\tau,\cdot)\|_H^p \mathrm{d}\tau \right)^{1/p} < \infty. \tag{1.47}$$

Endowed with the above norm, $L^p(0,T;H)$ is a Banach space. If $p = 2$, the norm (1.47) is induced by the inner product

$$(u,v)_{L^2(0,T;H)} = \int_0^T ((u(\tau,\cdot), v(\tau,\cdot))_H \,\mathrm{d}\tau \tag{1.48}$$

that makes $L^2(0,T;H)$ a Hilbert space.

We denote, for $1 \le p < \infty$, by $W^{1,p}(0,T;H)$ the Sobolev space of the functions $z \in L^p(0,T;H)$, whose weak derivative $\dot z \in L^p(0,T;H)$, with the norm

$$\|z\|_{W^{1,p}(0,T;H)} = \left(\int_0^T \|z(\tau,\cdot)\|_H^p \mathrm{d}\tau + \int_0^T \|\dot z(\tau,\cdot)\|_H^p \mathrm{d}\tau \right)^{1/p}, \tag{1.49}$$

these spaces are Banach spaces.

Chapter 2

A tracking problem in a battery model

This chapter is contained in

E. Hernández, C. Prieur, and E. Cerpa. A tracking problem for the state of charge in an electrochemical li-ion battery model. *Mathematical Control and Related Fields*, 2021. To appear.

Summary

2.1 Introduction

Today batteries are being developed to power a crescent and wide range of applications as laptops, smartphones, watches, electric vehicles, medicals devices and many others. Consequently, batteries are certainly in the middle of the technological development [3]. In this direction, li-ion batteries are gaining more and more attention due to its very good properties, compared to alternative battery technologies. For example, li-ion batteries provide one of the best energy-weight ratios and have a low self discharge when not in use [12].

An intelligent battery control system can ensure longevity and performance of battery, but such a type of improvements relies in an exhaustive understanding of energy storage. Thus, the modeling of li-on batteries has a key role in the design of battery management systems.

The literature on modeling of li-ion batteries is quite extensive. However, we can distinguished two groups of models. The first group is formed by equivalent circuit models (ECMs), which employ circuits elements to imitate the input-output behavior of a battery. The second group is formed by electrochemical models, which take into account electrochemical principles. Although electrochemical modeling approach has proved to have a better prediction capability compared to equivalent circuit models [12], the mathematical structure provides a huge challenge. The electro-chemical models arise many open questions in control. For instance, in [62] the author provides, in a brief way, a survey about the main challenges in battery management in which electro-chemical models are involved. For example, the problem to estimate the State of Charge, which indicates the stored energy at certain time and its time-evolution is also useful to determinate the health of the battery. Thus, control theory of electrochemical models needs important efforts from the control community.

This work aims at contributing in that direction. We are interested in studying the problem of tracking of the State of Charge in a battery modeled by an electrochemical model. In other words, given a reference State of Charge profile, we want to find the appropriate input current in order to get the real State of Charge near to the given reference. The model used along this work is called the Single Particle Model. For more details about its obtention, please see [62, 12].

The design of tracking controls has gained more and more attention. For instance, the growing demands on product quality and production efficiency, which require to turn away from the pure stabilization of an operating point towards tracking task as can be seen in some industrial applications, see for instance [58, 70, 18]. In that line, a potential application of the tracking of the State of Charge might be related with the dynamic pricing. For instance, for electrical vehicles, this means that the charging provider, which can be a distribution system operator or an operator/aggregator of charging stations, dynamically adapts the prices, which have to be payed by the final user for charging their electrical vehicles, see [51]. So the question of how to adapt the usage of the battery in order the operate at minimum cost could be solved to track an optimal State of Charge profile.

As it mentioned before, at the present section, the Single Particle Model (SPM), which is a reduction of the more general model due to Doyle-Fuller-Newman (DFN), is used to describe the behavior of a battery. The SPM considers each electrode as a single spherical particle and neglects the electrolyte dynamics, i.e., this model considers that lithium concentration in electrolyte phase remains constant. In the following we describe the SPM.

The spherical diffusion equation is used to describe the lithium concentration behavior $c^j(t,r)$ in solid phase. Thus, it holds

$$\begin{cases} c_t^j(t,r) = D^j\left[\frac{2}{r}c_r^j(t,r) + c_{rr}^j(t,r)\right], & (t,r)\in Q_j, \\ c_r^j(t,0) = 0, \quad c_r^j(t,R_j) = -j\rho_j I(t), & t\in(0,\infty), \\ c^j(0,r) = c_0^j(r), & r\in(0,R_j), \end{cases} \tag{2.1}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t}(t) = \phi_1(T_{amb} - T(t)) + \phi_2 I(t)V(t), \tag{2.2}$$

$$V(t) = \frac{RT(t)}{\alpha F}\sum_{j\in\{+,-\}} \sinh^{-1}(-j\omega_j(t)) + jU^j\left(c_s^j(t)\right) - R_f I(t), \tag{2.3}$$

where $j\in\{+,-\}$ indicates positive or negative electrode, D^j is the diffusivity, $Q^j = (0,\infty)\times(0,R_j)$ is the domain and R_j is the particle radius, $\rho_j = \frac{1}{D_j F a_j A L_j}$, ϕ_1, ϕ_2 are known parameters, $T(t)$ is temperature on the battery, $I(t)$ is the input current and $V(t)$ is the output voltage. The function $\omega_j(t)$ is given by

$$\omega_j(t) = \frac{I(t)}{\mu_j\sqrt{c_s^j(t)\left(c_{\max}^j - c_s^j(t)\right)}} \tag{2.4}$$

where $c_s^j(t) = c^j(t,R_j)$ is the surface concentration (or boundary concentration), $c_{\max}^j$ is the maximum ion concentration in the electrode j, $\mu_j = 2a^j AL^j\sqrt{c_e}$ are known parameters and U^j are equilibrium potentials of each electrode material. We detail all variables and parameters in Table 2.1.

Model states, inputs and outputs	
$c^{\pm}$	Lithium concentration in solid phase $[mol/m^3]$
$c_s(t)$	Lithium concentration at solid particle surface$[mol/m^2]$
c_e	Lithium concentration in electrolyte phase $[mol/m^3]$
T	Temperature $[K]$
I	Applied current, $[A/m^2]$
V	Output Voltage $[V]$
Electrochemical model parameters	
$D^{\pm}$	Diffusivity $[m^2/s]$
$R_{\pm}$	Particle radius in solid phase $[m]$
F	Faraday Constant $[C/mol]$
R	Universal gas constant $[J/mol \cdot K]$
α	Charge transfer coefficient $[-]$
$c^{\pm}_{\max}$	Maximum concentration of solid material $[mol/m^3]$
$U^{\pm}$	Open circuit potential of solid material $[V]$
R_f	Solid interphase films resistance $[\Omega \cdot m^2]$
$L^{\pm}$	Length of region $[m]$
A	Area $[m^2]$
ϕ_1	Heat transfer coefficient $[1/s]$
ϕ_2	Inverse of heat capacity $[J/K]^{-1}$
$\varepsilon^{\pm}$	Volume fraction of solid phase $[-]$

TABLE 2.1: Model variables and electrochemical parameters

A very precise formulation of the DFN model and the reduction to the SPM, can be found in [12].

The following definition establishes a precise formula for the State of Charge in a battery modeled by (2.1)-(2.3).

Definition 2.1.1. *Let $c^-(t,r)$ the Li-ion concentration in the negative electrode, then the battery State of Charge (SOC) is given by*

$$SOC(t) = \frac{3}{R_-^3 c^-_{\max}} \int_0^{R_-} c^-(t,r) r^2 \, dr. \tag{2.5}$$

For the sake of simplicity, it is defined the following non-dimensional variables,

$$\bar{r}_j = \frac{r}{R_j}, \quad \bar{t}_j = \frac{D_j}{R_j^2} t, \quad j \in \{+,-\}. \tag{2.6}$$

In order to keep a simple notation, the bars and j index on the space and time coordinate are removed. This normalization produces the following partial differential equation

$$\begin{cases} c^j_t(t,r) = \frac{2}{r} c^j_r(t,r) + c^j_{rr}(t,r), & (t,r) \in \bar{Q}, \\ c^j_r(t,0) = 0, \quad c^j_r(t,1) = -j\tilde{\rho}_j I(t), & t \in (0,\infty), \\ c^j(0,r) = c^j_0(r), & r \in (0,1), \end{cases} \tag{2.7}$$

where $\bar{Q} = (0,\infty) \times (0,1)$, $j \in \{+,-\}$ and $\tilde{\rho}_j = R_j \rho_j$. In order to precise the notation used, the initial condition $c^j_0(r)$ of the system (2.7) is the initial condition of the system (2.1) scaled to the domain $r \in (0,1)$.

After this normalization the State of Charge becomes

$$SOC(t) = \frac{3}{c_{\max}} \int_0^1 c(t,r) r^2 \, \mathrm{d}r. \tag{2.8}$$

We have removed the index $-$, in order to simplify the notation.

2.1.1 Problem statement and main results

In general words, the main objective of this work is to studying the problem of tracking the SOC to a reference trajectory denoted $SOC_{ref}(t)$. This problem has already been studied in a different context, as regulation problem for Parabolic PDE. See, on the one hand, [83] and [79], where the authors deal with the regulation problem with an internal P or PI control. On the other hand, in [73, 22, 23], the authors solve a tracking problem through a control acting on the boundary.

In this work, we deal with the regulation problem. To do that, we aim to apply the main idea of the certainty equivalence principle or separation principle, which refers to the fact that plug-in a convergent estimator in a stable closed-loop system does not change the stability. At this point, it is important to mention that, for the case of infinite-dimensional systems, the certainty equivalence principle may not apply as we know for the case of finite-dimensional systems. Early references recognize this difference. For example, one of the first contributions regarding the design of a finite-dimensional observer-based controller for PDEs was reported in [19]. In that paper, it is proved that under a number of suitable assumptions, a form of separation principle holds. In the same direction, in [4], referring to a Distributed Parameter System the author affirms that "There is no guarantee that a finite-dimensional controller can always produce closed-loop exponential stability with a given DPS." In that article it is proved, in the case of bounded input and bounded output, the stability of the resulting closed-loop system which was assessed for controllers with dimension large enough, but without providing an explicit criterion for the selection of the dimension parameter. Later, in [41] explicit conditions on the order of the finite-dimensional observer-based controller were reported. Another example is [78], where it is stated a certainty equivalence principle for a class of unstable-parabolic equations, more precisely

$$z_t = z_{xx} + \lambda z$$

for a large unknown parameter λ. In that paper, the authors provide a very specific update law $\dot{\hat{\lambda}}$, for the observer $\hat{\lambda}$, thus under that restrictive conditions the separation principle holds. Summarizing, in all these articles the stability of closed-loop systems were proven under very specific assumptions according to the particular cases in study.

The control design, developed along this chapter can be summarized as follows. We begin by dealing with the most simple case. We assume that we are able to measure the full state $c(t, r)$ of (2.7), then we design an output feedback control which achieves the tracking. After that, using the backstepping method, see for example [49, 82], we design an output feedback depending on the measure of the boundary concentration $c(t, 1)$. The next step in our design consists of replacing the boundary measure $c(t, 1)$ by a convergent observer, namely $\varphi(t)$.

As far as we know, in the literature this separation principle is used without proof. In [66] the authors propose an adaptive scheme to obtain $\varphi(t)$ based on the continuous Newton method. The proof of convergence of the scheme and of the closed-loop system is omitted. The authors of [64] state an exponential convergent scheme to obtain φ, but the proof of convergence of the system in closed loop is omitted.

In order to state the separation principle, we assume that this estimator $\varphi(t)$ satisfies the following assumption.

Assumption 2.1.2. *There exist a function $\varphi : [0, \infty) \to \mathbb{R}$ and positive constants L and μ such that*

$$|\varphi(t) - c(t, 1)| \leq Le^{-\mu t}, \quad \forall t \geq 0,$$

where $c(t, r)$ is the solution to (2.7).

Let us define, for some $p_1(r, \lambda)$ and $p_0(\lambda)$ (given later by the backstepping method), the following copy of the plant

$$\begin{cases} \partial_t \widehat{c_\varphi} = \frac{2}{r}\partial_r \widehat{c_\varphi} + \partial_{rr}\widehat{c_\varphi} + p_1(r, \lambda)(\varphi(t) - \widehat{c_\varphi}(t, 1)), \\ \partial_r \widehat{c_\varphi}(t, 0) = 0, \quad \partial_r \widehat{c_\varphi}(t, 1) = \tilde{\rho} I(t) + p_0(\lambda)(\varphi(t) - \widehat{c_\varphi}(t, 1)), \\ \widehat{c_\varphi}(0, r) = \widehat{c_\varphi}_0(r). \end{cases} \tag{2.9}$$

Our first result consists in the exponential stability of the observer error $\widetilde{c}(t, r) = c(t, r) - \widehat{c_\varphi}(t, r)$, which is stated in Theorem 2.1.3. This constitutes the main contribution of this chapter, which presents rigorous proofs of our statements on convergence.

Theorem 2.1.3. *Consider $\varphi : [0, \infty) \to \mathbb{R}$ and constants $L > 0$ and $\mu > 0$ satisfying Assumption 2.1.2, the initial condition $\widetilde{c}_0 = c_0(r) - \widehat{c_\varphi}_0(r)$ and the gains $p_0(\lambda)$ and $p_1(r, \lambda)$ given by*

$$p_0(\lambda) = \frac{\lambda}{2} \tag{2.10}$$

and

$$p_1(r, \lambda) = \left(\frac{\lambda}{(r^2 - 1)} + \frac{\lambda}{2}\right) J_2\left(\sqrt{\lambda(r^2 - 1)}\right) - \frac{\lambda}{2} J_0\left(\sqrt{\lambda(r^2 - 1)}\right), \tag{2.11}$$

where J_0 and J_2 are the zero and second order Bessel functions of first kind respectively.

Therefore there exists $\lambda_{\sup} > 2 + \sqrt{6}$ such that for all $\lambda \in [2 + \sqrt{6}, \lambda_{\sup})$ the function $\tau(\lambda)$ defined by

$$\tau(\lambda) = \frac{\pi^2}{2} - \frac{2}{\lambda}\|p_1(\cdot, \lambda)\|^2_{L^2_r(0,1)} \tag{2.12}$$

is positive. Moreover, depending on μ, the L^2_r norm of the observer error $\widetilde{c}(t, r)$ satisfies one of the following cases:

1. *If $\mu > \frac{\tau(\lambda)}{2}$ for all $\lambda \in [2 + \sqrt{6}, \lambda_{\sup})$, then*

$$\|\widetilde{c}(t, \cdot)\|^2_{L^2_r(0,1)} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\lambda^3 + 4\lambda)}{2|\tau(\lambda) - 2\mu|}\right) e^{-\tau(\lambda)t}, \quad \forall t \geq 0, \forall \lambda \in [2 + \sqrt{6}, \lambda_{\sup}). \tag{2.13}$$

2. *If $\mu = \frac{\tau(\bar{\lambda})}{2}$, for some $\bar{\lambda} \in [2 + \sqrt{6}, \lambda_{\sup})$, then*

$$\|\widetilde{c}(t, \cdot)\|^2_{L^2_r(0,1)} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\bar{\lambda}^3 + 4\bar{\lambda})}{2} t\right) e^{-\tau(\bar{\lambda})t}, \quad \forall t \geq 0. \tag{2.14}$$

The proof of this theorem can be found in Section 2.4.2.

Remark 2.1.4. *In the Theorem 2.1.3 as well as in its proof, see Section 2.4.6, we consider* $[2+\sqrt{6}, \lambda_{\text{sup}})$ *as the biggest interval in which* $\tau(\lambda) > 0$ *for all* $\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$.

The followings results are a direct consequence of Theorem 2.1.3 and describe the performance of observer $\widehat{c}_\varphi(t,r)$.

Corollary 2.1.5. *Let* $\lambda^* = 2+\sqrt{6}$. *Depending on* μ *we have the following*

1. *if* $2\mu > \tau(\lambda^*)$, *then the highest decay rate of* $\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)}$ *is* $\tau(\lambda^*)$ *and the transient state is bounded. Moreover, it holds*

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)} \le 2\|\widetilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\lambda^{*3}+4\lambda^*)}{2|\tau(\lambda^*)-2\mu|}, \quad \forall t \ge 0, \tag{2.15}$$

2. *if* $2\mu \le \tau(\lambda^*)$, *then the decay ratio of* $\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)}$ *is* 2μ *and the transient state is bounded. Moreover, it holds*

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)} \le \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2\tau(\bar{\lambda})} \exp\left\{ \frac{4\|\widetilde{c}_0\|^2_{L^2_r(0,1)}\tau(\bar{\lambda})}{L^2(\bar{\lambda}^3+4\bar{\lambda})} - 1 \right\}, \quad \forall t \ge 0, \tag{2.16}$$

where $\bar{\lambda}$ *is solution to equation* $2\mu = \tau(\bar{\lambda})$.

Let us define the following

$$N_1(\lambda) = 2\|\widetilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}.$$

Corollary 2.1.6. *Let* $\lambda^* = 2+\sqrt{6}$ *and* $[\lambda^*, \lambda_{\text{sup}}]$ *the interval given by Theorem 2.1.3. If* $2\mu > \tau(\lambda^*)$, *then*

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r(0,1)} \le 2\|\widetilde{c}_0\|^2_{L^2_r(0,1)} + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2|\tau(\bar{\lambda})-2\mu|}, \quad \forall t \ge 0, \tag{2.17}$$

where $\bar{\lambda} = \underset{\lambda\in[\lambda^*,\lambda_{\text{sup}}]}{\arg\min}\ N_1(\lambda)$ *and the decay ratio is given by* $\tau(\bar{\lambda})$.

From the previous exponential stability result for $\widetilde{c}$ stated in Theorem 2.1.3, we are able to prove the following result.

Theorem 2.1.7. *Consider* $\varphi : [0,\infty) \to \mathbb{R}$ *and constants* $L > 0$ *and* $\mu > 0$ *satisfying Assumption 2.1.2, gains* $p_0(\lambda)$ *and* $p_1(r,\lambda)$ *given by (2.10) and (2.11) respectively. There exists* $\lambda_{\text{sup}} > 2+\sqrt{6}$ *such that for* $\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$ *we define the input current*

$$I(t) = \frac{c_{\max}}{3\tilde{\rho}}\left(\dot{SOC}_{ref}(t) + \gamma\left(SOC_{ref}(t) - \widehat{SOC}_\varphi(t)\right)\right), \tag{2.18}$$

where

$$\widehat{SOC}_\varphi(t) = \frac{3}{c_{\max}}\int_0^1 \widehat{c}_\varphi(t,r)r^2\,dr,$$

$\gamma > 0$ *is a design parameter and* $\widehat{c}_\varphi(t,r)$ *is the solution to (2.9). This feedback control* $I(t)$ *forces the system to satisfy*

$$|SOC_{ref}(t) - SOC(t)| \to 0, \quad t \to \infty \tag{2.19}$$

with an exponential rate, depending on the parameters.

2.1.2 Organization

The remaining part of this chapter is organized as follows. In Section 2.2 we design an input $I(t)$ which depends on full state measurements of the concentration. In Section 2.3 we improve the previous design of $I(t)$ by considering partial state measurements of the concentration on the boundary. Section 2.4 is finally dedicated to the design of the current input $I(t)$ used in Theorem 2.1.7. Being precise, in Section 2.4.2 and Section 2.4.6 we provide the proof of Theorem 2.1.3 and Theorem 2.1.7 respectively. In Section 2.5 we illustrate the results by some numerical simulations.

2.2 Tracking with a full state feedback

The following proposition give us a starting point for the design of an input current in order to regulate the $SOC(t)$.

Proposition 2.2.1. *Consider system (2.7), the State of Charge $SOC(t)$ defined by the equation (2.8) and the reference trajectory $SOC_{ref}(t)$. Let the input current be*

$$I(t) = \frac{c_{max}}{3\tilde{\rho}} \left(S\dot{O}C_{ref}(t) + \gamma \left(SOC_{ref}(t) - SOC(t) \right) \right), \tag{2.20}$$

where $\gamma > 0$ is a constant design parameter. Then, there exists a constant $C > 0$ such that for all $t > 0$

$$|SOC_{ref}(t) - SOC(t)| \leq Ce^{-\gamma t}. \tag{2.21}$$

Proof. See Section A.1 in the Appendix. □

Notice that this input $I(t)$ depends on full state $c(t, r)$ (see definition of SOC given by (2.8)) and this is a state feedback law. However, in most cases we have no access to the full state of the system. Thus, it is more realistic to design an output feedback which only depends on some partial measure of the state. This is the goal of next section.

2.3 Tracking from partial state measurements

We design a feedback control which depends on a partial measurement of the state given by the boundary concentration. To do that, we employ the Backstepping method (see for instance [49, 64, 65, 66]).

2.3.1 State observer

We define the anode state observer structure, which consists in a copy of (2.7) plus a boundary state error injection, as follows

$$\begin{cases} \widehat{c}_t(t, r) = \frac{2}{r}\widehat{c}_r + \widehat{c}_{rr} + p_1(r, \lambda)\widetilde{c}(t, 1), & (t, r) \in \bar{Q}, \\ \widehat{c}_r(t, 0) = 0, \quad \widehat{c}_r(t, 1) = \tilde{\rho} I(t) + p_0(\lambda)\widetilde{c}(t, 1), & t \in (0, \infty), \\ \widehat{c}(0, r) = \widehat{c}_0(r), & r \in (0, 1), \end{cases} \tag{2.22}$$

where $\widetilde{c}(t, 1) = c(t, 1) - \widehat{c}(t, 1)$, λ is a design parameter and $p_0(\lambda)$, $p_1(r, \lambda)$ are tuning gains to be chosen later.

Remark 2.3.1. *The observer $\widehat{c}(t,r)$ requires the measure of $c(t,1)$. The gains $p_0(\lambda)$ and $p_1(r,\lambda)$ have to be determinated in the way of ensure the convergence of the observer to the real state.*

The estimation error $\widetilde{c}(t,r) = c(t,r) - \hat{c}(t,r)$ follows the dynamics

$$\begin{cases} \widetilde{c}_t(t,r) = \frac{2}{r}\widetilde{c}_r + \widetilde{c}_{rr} - p_1(r,\lambda)\widetilde{c}(t,1), & (t,r) \in \bar{Q}, \\ \widetilde{c}_r(t,0) = 0, \quad \widetilde{c}_r(t,1) + p_0(\lambda)\widetilde{c}(t,1) = 0, & t \in (0,\infty), \\ \widetilde{c}(0,r) = \widetilde{c}_0(r), & r \in (0,1), \end{cases} \tag{2.23}$$

with $\widetilde{c}_0(r) = c_0 - \widehat{c}_0$. We search for a kernel $p(r,s)$ such that the following transformation

$$\widetilde{c}(t,r) = \widetilde{z}(t,r) - \int_r^1 p(r,s)\widetilde{z}(s)\,\mathrm{d}s \tag{2.24}$$

is well-defined and where $\widetilde{z}$ is the solution to the following well-posed target system

$$\begin{cases} \widetilde{z}_t = \frac{2}{r}\widetilde{z}_r + \widetilde{z}_{rr} - \lambda\widetilde{z}, & (t,r) \in \bar{Q}, \\ \widetilde{z}_r(t,0) = 0, \quad \widetilde{z}_r(t,1) = 0, & t \in (0,\infty), \\ \widetilde{z}(0,r) = \widetilde{z}_0(r), & r \in (0,1). \end{cases} \tag{2.25}$$

Remark 2.3.2. *The choice of the target system is a crucial part of the Backstepping method. The main idea behind this method consists in deducing the exponential stability property of the error system from of that property for the target system.*

For the sake of completeness, we include the next result for the heat equation ensuring the well-posedness and the exponential stability in $H_r^1(0,1)$ norm of the target system (2.25).

Proposition 2.3.3. *Let $\lambda > 0$. For all initial condition $\widetilde{z}_0 \in H_r^1(0,1)$, there exists an unique $\widetilde{z} \in C([0,\infty); H_r^1(0,1)) \cap C^1([0,\infty); L_r^2(0,1))$ solution to (2.25). Moreover, we get the estimation*

$$\|\widetilde{z}(t,\cdot)\|^2_{H_r^1(0,1)} \leq e^{-2\lambda t}\|\widetilde{z}_0\|^2_{H_r^1(0,1)}, \quad \forall t \geq 0. \tag{2.26}$$

Proof. See Section A.2 in the Appendix. □

Let us use the integral transformation (2.24) and the systems (2.23) and (2.25). After some calculations, see for instance [49, Chapter 4], we get the following system for the kernel $p(r,s)$

$$\begin{cases} p_{rr} + \frac{2}{r}p_r + 2\left(\frac{p}{s}\right)_s - p_{ss} = -\lambda p(r,s), & (r,s) \in T, \\ p(r,r) = -\frac{\lambda}{2}r, \quad p(0,r) = 0, & r \in (0,1), \end{cases} \tag{2.27}$$

where $T = \{(r,s) \in \mathbb{R}^2 : \ 0 \leq r \leq s \leq 1\}$.

The gain equations for the anode observer (2.22) are defined by

$$p_0(\lambda) = \frac{\lambda}{2}, \tag{2.28}$$

$$p_1(r,\lambda) = 2p(r,1) - p_s(r,1), \quad \forall r \in (0,1), \tag{2.29}$$

where $p(r,s)$ is the solution to (2.27).

The well-posedness of the kernel equation (2.27) was studied in [82]. The following lemma explains how to solve the kernel equation and then, how to get the gain observer $p_1(r,\lambda)$ explicitly.

Lemma 2.3.4. *Let $\lambda > 0$ in the target system (2.25). The solution to (2.27) is given by*

$$p(r,s) = -\lambda s \frac{J_1\left(\sqrt{\lambda(r^2-s^2)}\right)}{\left(\sqrt{\lambda(r^2-s^2)}\right)}, \tag{2.30}$$

where J_1 is the first order Bessel function of first kind. Moreover the gain $p_1(r,\lambda)$ is given by

$$p_1(r,\lambda) = \left(\frac{\lambda}{(r^2-1)} + \frac{\lambda}{2}\right) J_2\left(\sqrt{\lambda(r^2-1)}\right) - \frac{\lambda}{2} J_0\left(\sqrt{\lambda(r^2-1)}\right), \tag{2.31}$$

Proof. See Section A.3 in the Appendix. □

Now, we prove the exponential decay in $H_r^1(0,1)$ norm of the error (2.23). We define the following operator

$$\begin{aligned} \Lambda : H_r^1(0,1) &\longrightarrow H_r^1(0,1) \\ \widetilde{c} &\longmapsto \Lambda(\widetilde{c}) = \tilde{c} + \int_r^1 l(r,s)\widetilde{c}(s)\,\mathrm{d}s. \end{aligned}$$

The operator Λ is the inverse transformation of (2.24) and is well defined, linear and continuous. To see that, it is important to notice that the l-kernel associated to λ is minus the p-kernel associated to $-\lambda$, as explained in [49, Chapter 4].

The next proposition allows to infer the exponential stability property of the error system (2.23) from the target system (2.25).

Proposition 2.3.5. *For all $\lambda > 0$, there exists a constant $M > 1$ such that the error system (2.23), with the gains $p_0(\lambda)$ and $p_1(r,\lambda)$ defined by the equations (2.10) and (2.11) respectively, satisfies for all $t \geq 0$,*

$$\|\widetilde{c}(t,\cdot)\|_{H_r^1(0,1)} \leq M e^{-\lambda t} \|\widetilde{c}_0\|_{H_r^1(0,1)}. \tag{2.32}$$

Proof. The map Λ is a linear continuous operator with a continuity constant greater than one. Indeed, this follows from the fact that the first term in (2.24) is the identity. Also Λ is invertible. Thus, the same properties hold for Λ^{-1} (thanks to the Open Map Theorem, see [7, Corollary 2.7]). Using the exponential stability of the target system we have the following inequality, for all $t \geq 0$,

$$\|\widetilde{c}(t,\cdot)\|_{H_r^1(0,1)} \leq M e^{-\lambda t} \|\widetilde{c}_0\|_{H_r^1(0,1)}, \tag{2.33}$$

where $\tilde{c}_0$ is the initial condition of the system (2.23). □

2.3.2 State of Charge Estimator

From (2.8), an appropriate estimator for the State of Charge is

$$\widehat{SOC}(t) = \frac{3}{c_{\max}} \int_0^1 \widehat{c}(t,r) r^2 \,\mathrm{d}r, \tag{2.34}$$

where $\widehat{c}(t,r)$ is solution to the observer system (2.22). This estimator of the State of Charge was used in [65, 66, 64].

Proposition 2.3.6. *Consider the State of Charge estimator defined by* (2.34)*. If* $\lambda > 0$*, then*

$$|SOC(t) - \widehat{SOC}(t)| \leq \frac{\sqrt{3}M}{c_{\max}} e^{-\lambda t} \|\tilde{c}_0\|_{H^1_r(0,1)}, \quad \forall t \geq 0. \tag{2.35}$$

Proof. We consider the estimation error for State of Charge

$$SOC(t) - \widehat{SOC}(t) = \frac{3}{c_{\max}} \int_0^1 \tilde{c}(t,r) r^2 \,\mathrm{d}r. \tag{2.36}$$

Now, by Cauchy-Schwartz inequality and Proposition 2.3.5, we obtain the following inequality, for all $t \geq 0$,

$$|SOC(t) - \widehat{SOC}(t)| \leq \frac{\sqrt{3}M}{c_{\max}} e^{-\lambda t} \|\tilde{c}_0\|_{H^1_r(0,1)}, \tag{2.37}$$

that proves Proposition 2.3.6. □

Remark 2.3.7. *Notice that* $\widehat{SOC}(t)$ *is an observer of* $SOC(t)$ *which depends only on a partial measurement of the full state* $c(t,r)$*. Being precise,* $\widehat{SOC}(t)$ *just depends on the boundary concentration* $c(t,1)$*.*

2.3.3 Tracking of the SOC

In the following we prove that the feedback control (2.20) works even if we replace $SOC(t)$ by $\widehat{SOC}(t)$.

Theorem 2.3.8. *Consider the system* (2.7)*,* $\lambda > 0$ *and* $\widehat{SOC}(t)$ *defined by* (2.34)*. If the input current* $I(t)$ *is selected as following*

$$I(t) = \frac{c_{\max}}{3\tilde{\rho}} \left(\dot{SOC}_{ref}(t) + \gamma \left(SOC_{ref}(t) - \widehat{SOC}(t) \right) \right) \tag{2.38}$$

where $\gamma > 0$ *is a design parameter. Then there exist three cases depending on* γ

1. *If* $\gamma < 2\lambda$*, then*

$$(SOC_{ref}(t) - SOC(t))^2 \leq \left((SOC_{ref}(0) - SOC(0))^2 + \frac{3\gamma M^2 \|\tilde{c}_0\|^2_{H^1_r(0,1)}}{2c^2_{\max}|\gamma - 2\lambda|} \right) e^{-\gamma t}, \quad \forall t \geq 0. \tag{2.39}$$

2. *If* $\gamma = 2\lambda$*, then*

$$(SOC_{ref}(t) - SOC(t))^2 \leq \left((SOC_{ref}(0) - SOC(0))^2 + \frac{3\gamma M^2 \|\tilde{c}_0\|^2_{H^1_r(0,1)}}{2c^2_{\max}} t \right) e^{-\gamma t}, \quad \forall t \geq 0. \tag{2.40}$$

3. *If* $\gamma > 2\lambda$*, then*

$$(SOC_{ref}(t) - SOC(t))^2 \leq \left((SOC_{ref}(0) - SOC(0))^2 + \frac{3\gamma M^2 \|\widetilde{c}_0\|^2_{H^1_r(0,1)}}{2c^2_{\max}(\gamma - 2\lambda)} \right) e^{-2\lambda t}, \quad t \geq 0. \tag{2.41}$$

Proof. See Section A.4 in the Appendix. □

Remark 2.3.9. *We have proved that the input current given by (2.38) achieves a regulation of the State of Charge of the system using only the partial measurement $c(t,1)$. Moreover, this regulation has an exponential convergence ratio. This is an improvement taking account the previous design proposed in Section 2.2. However, this design might be still being considered unrealistic, in the sense to require an online measure of the boundary concentration. In order to avoid this assumption, in the next section we propose a design which uses a convergent estimator of the boundary concentration $c(t,1)$.*

2.4 Tracking from a convergent estimator and proofs of the main results

Along this section we focus on the design of a regulator which solves the problem of the tracking of SOC using a convergent estimator of the boundary concentration $c(t,1)$. We provide a proof of the main results of this work namely, Theorem 2.1.3, Corollaries 2.1.5 and 2.1.6 and Theorem 2.1.7 successively.

2.4.1 Observer design for the ion concentration

In this subsection, we do not assume that we measure $c(t,1)$ (the real surface concentration in the negative electrode). We use instead an estimator $\varphi(t)$. As we mentioned in Section 2.1.1. We assume Assumption 2.1.2 on $\varphi(t)$ and $c(t,1)$.

We define a new observer equation in which we have replaced the surface concentration $c(t,1)$ by the estimation $\varphi(t)$ and we get

$$\begin{cases} \partial_t \widehat{c_\varphi}(t,r) = \frac{2}{r}\partial_r \widehat{c_\varphi} + \partial_{rr}\widehat{c_\varphi} + p_1(r,\lambda)(\varphi(t) - \widehat{c_\varphi}(t,1)), \\ \partial_r \widehat{c_\varphi}(t,0) = 0, \quad \partial_r \widehat{c_\varphi}(t,1) = \tilde{\rho} I(t) + p_0(\lambda)(\varphi(t) - \widehat{c_\varphi}(t,1)), \\ \widehat{c_\varphi}(0,r) = \widehat{c_\varphi}_0(r), \end{cases} \tag{2.42}$$

where the gains $p_0(\lambda)$ and $p_1(r,\lambda)$ are still defined by (2.10) and (2.11), respectively. In the following subsection we give conditions for the convergence of the observer error $\widetilde{c}(t,r) = c(t,r) - \widehat{c_\varphi}(t,r)$.

We define the surface concentration estimation error by $\eta(t) = \varphi(t) - c(t,1)$. Using the state equations (2.7) and the observer equations (2.42) we obtain the following system for the error.

$$\begin{cases} \widetilde{c}_t(t,r) - \frac{2}{r}\widetilde{c}_r - \widetilde{c}_{rr} + p_1(r,\lambda)\widetilde{c}(t,1) = -p_1(r,\lambda)\eta(t), \\ \widetilde{c}_r(t,0) = 0, \quad \widetilde{c}_r(t,1) + p_0(\lambda)\widetilde{c}(t,1) = -p_0(\lambda)\eta(t), \\ \widetilde{c}(0,r) = \widetilde{c}_0(r). \end{cases} \tag{2.43}$$

Remark 2.4.1. *The system (2.43) can be seen as the system (2.23) with the perturbation terms $p_1(r,\lambda)\eta(t)$ in the domain and $p_0(\lambda)\eta(t)$ on the boundary, respectively.*

2.4.2 Proof of Theorem 2.1.3

Here we prove Theorem 2.1.3 which gives the conditions for the convergence of the observer error.

Consider $\widetilde{c}(t,r) = u(t,r) + v(t,r)$, where u is solution of the following system

$$\begin{cases} u_t - \frac{2}{r}u_r - u_{rr} + p_1(r,\lambda)u(t,1) = 0, \\ u_r(t,0) = 0, \quad u_r(t,1) + p_0(\lambda)u(t,1) = -p_0(\lambda)\eta(t), \\ u(0,r) = \widetilde{c}_0(r), \end{cases} \tag{2.44}$$

and v is solution of the following system

$$\begin{cases} v_t - \frac{2}{r}v_r - v_{rr} + p_1(r,\lambda)v(t,1) = -p_1(r,\lambda)\eta(t), \\ v_r(t,0) = 0, \quad v_r(t,1) + p_0(\lambda)v(t,1) = 0, \\ v(0,r) = 0. \end{cases} \tag{2.45}$$

The main idea is to prove the exponential decay of the norm for u and v. We begin by multiplying the first line of (2.44) by $u(t,r)r^2$ and then we integrate over $r \in (0,1)$. Using the boundary conditions in (2.44), we obtain, for all $t \geq 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u^2r^2\,\mathrm{d}r + \int_0^1 u_r^2r^2\,\mathrm{d}r + p_0(\lambda)u^2(1) = \\ - p_0(\lambda)u(1)\eta(t) - u(1)\int_0^1 p_1(r,\lambda)ur^2\,\mathrm{d}r. \tag{2.46}$$

On the righthand side of the above equality, we apply the Cauchy-Schwartz inequality and two times the Young inequality. Consequently, we obtain, for all $\delta > 0$, $\beta > 0$ and $t \geq 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \|u_r\|^2_{L^2_r} + p_0(\lambda)u^2(1) \leq \\ \left(\frac{\delta p_0(\lambda)}{2} + \frac{\beta}{2}\right)u^2(1) + \frac{1}{2\delta}p_0(\lambda)\eta^2(t) + \frac{1}{2\beta}\|p_1(r,\lambda)\|^2_{L^2_r}\|u\|^2_{L^2_r}. \tag{2.47}$$

Recall the following version of the Poincaré inequality for the lefthand side of (2.47)

$$\int_0^1 w^2r^2\,\mathrm{d}r \leq \frac{4}{\pi^2}w^2(1) + \frac{4}{\pi^2}\int_0^1 w_r^2r^2\,\mathrm{d}r. \tag{2.48}$$

Then, for all $\delta > 0$, $\beta > 0$ and $t \geq 0$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \frac{\pi^2}{4}\|u\|^2_{L^2_r} + (p_0(\lambda) - 1)\,u^2(1) \leq \\ \left(\frac{\delta p_0(\lambda)}{2} + \frac{\beta}{2}\right)u^2(1) + \frac{1}{2\delta}p_0(\lambda)\eta^2(t) + \frac{1}{2\beta}\|p_1(r,\lambda)\|^2_{L^2_r}\|u\|^2_{L^2_r}. \tag{2.49}$$

Rearranging terms in previous inequality we get, for all $\delta > 0$, $\beta > 0$ and $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \left(\frac{\pi^2}{2} - \frac{1}{\beta}\|p_1(r,\lambda)\|^2_{L^2_r}\right)\|u\|^2_{L^2_r}$$

$$+ \left(2p_0(\lambda) - 2 - \delta p_0(\lambda) - \beta\right) u^2(1) \leq \frac{1}{\delta} p_0(\lambda)\eta^2(t). \quad (2.50)$$

We recall the definition of the gain $p_0(\lambda) = \frac{\lambda}{2}$, given by (2.10). Let us set $\delta = \frac{2}{\lambda^2}$ and $\beta = \frac{\lambda}{2}$. Then, from the previous inequality, we obtain for all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \left(\frac{\pi^2}{2} - \frac{2}{\lambda}\|p_1(r,\lambda)\|^2_{L^2_r}\right)\|u\|^2_{L^2_r} + \left(\frac{\lambda}{2} - 2 - \frac{1}{\lambda}\right)u^2(1) \leq \frac{\lambda^3}{4}\eta^2(t). \quad (2.51)$$

We define the following function which will be helpful to study (2.51),

$$\tau(\lambda) = \frac{\pi^2}{2} - \frac{2}{\lambda}\|p_1(\cdot,\lambda)\|^2_{L^2_r}, \quad (2.52)$$

where $p_1(r,\lambda)$ is given by (2.11).

It is possible to find an interval J such that $\tau(\lambda) > 0$ and $\frac{\lambda}{2} - 2 - \frac{1}{\lambda} \geq 0$, for all $\lambda \in J$. Indeed, on the one hand $\frac{\lambda}{2} - 2 - \frac{1}{\lambda} \geq 0$, for all $\lambda \geq 2 + \sqrt{6}$. On the other hand we check that $\tau(2+\sqrt{6}) > 0$ and since τ is a continuous function, there exists such interval.

Let us consider J as the biggest interval of the form $[2+\sqrt{6}, \lambda_{\text{sup}})$ such that $\tau(\lambda) > 0$, for all $\lambda \in J$. See Figure 2.1 for an example.

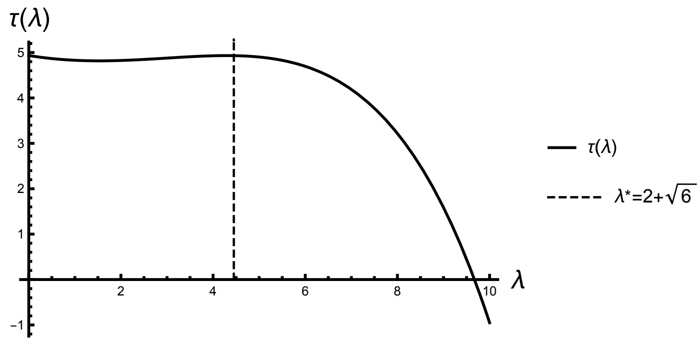


FIGURE 2.1: The continuous function τ is positive in $[2+\sqrt{6}, \lambda_{\text{sup}})$.

Under those conditions over parameter λ we obtain from (2.51), for all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \tau(\lambda)\|u\|^2_{L^2_r} + \leq \frac{\lambda^3}{4}\eta^2(t). \quad (2.53)$$

Multiplying (2.53) by $e^{\tau(\lambda)t}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{L^2_r} e^{\tau(\lambda)t}\right) \leq \frac{\lambda^3}{4}\eta^2(t)e^{\tau(\lambda)t}$$

We recall Assumption 2.1.2 on the estimator $\varphi(t)$. Taking account this, and the above inequality, we get, for all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{L^2_r} e^{\tau(\lambda)t}\right) \leq \frac{\lambda^3 L^2}{4} e^{(\tau(\lambda)-2\mu)t}. \quad (2.54)$$

We distinguish two cases:

1. Assume $\mu > \frac{\tau(\lambda)}{2}$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$.

Integrating inequality (2.54) over $(0,t)$, we get for all $t \geq 0$

$$\begin{aligned}\|u\|^2_{L^2_r} e^{\tau(\lambda)t} &\leq \|u_0\|^2_{L^2_r} + \frac{L^2\lambda^3}{4(\tau(\lambda)-2\mu)}\left(e^{(\tau(\lambda)-2\mu)t}-1\right)\\ &\leq \|u_0\|^2_{L^2_r} + \frac{L^2\lambda^3}{4|\tau(\lambda)-2\mu|}.\end{aligned}$$

This implies that, for all $t \geq 0$,

$$\|u\|^2_{L^2_r} \leq \left(\|u_0\|^2_{L^2_r} + \frac{L^2\lambda^3}{4|\tau(\lambda)-2\mu|}\right) e^{-\tau(\lambda)t}. \tag{2.55}$$

2. Assume $\mu = \frac{\tau(\bar{\lambda})}{2}$, for some $\bar{\lambda} \in [2+\sqrt{6}, \lambda_{\sup})$.
From (2.54), integrating over $(0,t)$, it holds, for all $t \geq 0$

$$\|u\|^2_{L^2_r} e^{\tau(\bar{\lambda})t} \leq \|u_0\|^2_{L^2_r} + \frac{L^2\bar{\lambda}^3}{4}t.$$

Therefore, for all $t \geq 0$

$$\|u\|^2_{L^2_r} \leq \left(\|u_0\|^2_{L^2_r} + \frac{L^2\bar{\lambda}^3}{4}t\right) e^{-\tau(\bar{\lambda})t}. \tag{2.56}$$

Now is turn to get an estimate of the L^2_r norm of v. Similar as before, let us consider the first line in (2.45). Multiplying by $v(t,r)r^2$ and then perform an integration by parts in $r \in (0,1)$, and using the boundary conditions in (2.45), we get for all $t \geq 0$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2_r} + \|v_r\|^2_{L^2_r} + p_0(\lambda)v^2(1) =\\ &\qquad - v(1)\int_0^1 p_1(r,\lambda)vr^2\,\mathrm{d}r - \eta(t)\int_0^1 p_1(r,\lambda)vr^2\,\mathrm{d}r.\end{aligned} \tag{2.57}$$

As before, on the righthand side of (2.57), we use the Cauchy-Schwartz inequality and two times the Young inequality, with the definition of $p_0(\lambda)$ given by (2.10), to get, for all $t \geq 0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2_r} + \|v_r\|^2_{L^2_r} + \frac{\lambda}{2}v^2(1) \leq \frac{\lambda}{2}v^2(1) + \frac{\lambda}{2}\eta^2(t) + \frac{1}{\lambda}\|p_1(r,\lambda)\|^2_{L^2_r}\|v\|^2_{L^2_r}. \tag{2.58}$$

Applying the Poincaré inequality on the lefthand side of (2.58) and rearranging terms we obtain, for all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2_r} + \tau(\lambda)\|v\|^2_{L^2_r} \leq \lambda\eta^2(t). \tag{2.59}$$

Multiplying the above inequality by $e^{\tau(\lambda)t}$ and taking account the Assumption 2.1.2 over the estimator φ it holds, for all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\tau(\lambda)t}\|v\|^2_{L^2_r}\right) \leq L^2\lambda e^{(\tau(\lambda)-2\mu)t}. \tag{2.60}$$

As before, we distinguish two cases. We omit the computations in reason of its similarities with the computations to obtain the L^2_r norm estimation for u.

If $\mu > \frac{\tau(\lambda)}{2}$ for all $\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$. Then it holds

$$\|v\|^2_{L^2_r} \leq \frac{L^2\lambda}{|\tau(\lambda) - 2\mu|} e^{-\tau(\lambda)t}, \quad \forall t \geq 0. \tag{2.61}$$

If $\mu = \frac{\tau(\bar\lambda)}{2}$, for some $\bar\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$. Then we obtain

$$\|v\|^2_{L^2_r} \leq L^2\bar\lambda t e^{-\tau(\bar\lambda)t}, \quad \forall t \geq 0. \tag{2.62}$$

We recall in that $v(0,r) = 0$.

Finally, collecting inequalities, (2.55), (2.56), (2.61) and (2.62) and using $\|c\|^2_{L^2_r} = \|u+v\|^2_{L^2_r} \leq 2(\|u\|^2_{L^2_r} + \|v\|^2_{L^2_r})$, we conclude that

1. If $\mu > \frac{\tau(\lambda)}{2}$ for all $\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$, then

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right) e^{-\tau(\lambda)t}, \quad \forall t \geq 0, \tag{2.63}$$

2. If $\mu = \frac{\tau(\bar\lambda)}{2}$, for some $\bar\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$, then

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar\lambda^3+4\bar\lambda)}{2} t\right) e^{-\tau(\bar\lambda)t}, \quad \forall t \geq 0. \tag{2.64}$$

The proof of Theorem 2.1.3 is complete. □

2.4.3 Proof of Corollary 2.1.5

Let $\lambda^* = 2+\sqrt{6}$. Note that the function τ is a decreasing continuous function on $[\lambda^*, \lambda_{\text{sup}})$. So its maximum is attained at $\tau(\lambda^*)$.

Let $2\mu > \tau(\lambda^*)$ then $2\mu \geq \tau(\lambda)$, for all $\lambda \in [\lambda^*, \lambda_{\text{sup}})$, so in virtue of Theorem 2.1.3 it holds

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right) e^{-\tau(\lambda)t}, \quad \forall t \geq 0, \forall\lambda \in [\lambda^*, \lambda_{\text{sup}}). \tag{2.65}$$

From the above inequality it is easy see that the fastest decayment for $\|\widetilde{c}(\cdot,t)\|^2_{L^2_r}$ is achieved if $\tau(\lambda) = \tau(\lambda^*)$ and since that $e^{-\tau(\lambda^*)t} < 1$, for all $t \geq 0$, we obtain an estimation of the transient state of $\|\widetilde{c}(t,\cdot)\|^2_{L^2_r}$ given by

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r} \leq 2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^{*3}+4\lambda^*)}{2|\tau(\lambda^*)-2\mu|}, \quad \forall t \geq 0. \tag{2.66}$$

If $2\mu \leq \tau(\lambda^*)$, by the monotonicity of τ on the interval $[\lambda^*, \lambda_{\text{sup}})$ there exists $\bar\lambda$ such that $2\mu = \tau(\bar\lambda)$. Now, the Theorem 2.1.3, it holds that

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar\lambda^3+4\bar\lambda)}{2} t\right) e^{-\tau(\bar\lambda)t}, \quad \forall t \geq 0. \tag{2.67}$$

Then, decay rate is given by $\tau(\bar\lambda) = 2\mu$.

Let us consider $\bar\lambda$ fix and we define $N_{\bar\lambda}(t) = \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar\lambda^3+4\bar\lambda)}{2} t\right) e^{-\tau(\bar\lambda)t}$. It is not difficult to see that $N_{\bar\lambda}(t)$ reaches its maximum at $t^* = \frac{1}{\tau(\bar\lambda)} - \frac{2\|\widetilde{c}_0\|^2_{L^2_r}}{L^2(\bar\lambda^3+4\bar\lambda)}$. It follows with $e^{-\tau(\bar{\lambda})t^*} < 1$, that

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r} \leq \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2\tau(\bar{\lambda})} \exp\left\{\frac{4\|\widetilde{c}_0\|^2_{L^2_r}\tau(\bar{\lambda})}{L^2(\bar{\lambda}^3+4\bar{\lambda})} - 1\right\}, \quad \forall t \geq 0. \tag{2.68}$$

The proof of Corollary 1.2.3 is complete. □

2.4.4 Proof of Corollary 2.1.6

Since that $2\mu > \tau(\lambda)$, by Theorem 2.1.3, it holds

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right) e^{-\tau(\lambda)t}, \quad \forall t \geq 0, \forall \lambda \in [\lambda^*, \lambda_{\sup}). \tag{2.69}$$

On the other hand, $\tau(\lambda) - \mu \neq 0$, for all $\lambda \in [\lambda^*, \lambda_{\sup}]$ then $N_1(\lambda)$ is a continuous function defined on a compact interval, in consequence there exists $\bar{\lambda}$ such that, $N_1(\lambda) \geq N_1(\bar{\lambda})$, for all $\lambda \in [\lambda^*, \lambda_{\sup}]$.

Let $\bar{\lambda}$ such that minimizes $N_1(\lambda)$, then it holds

$$\|\widetilde{c}(t,\cdot)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2|\tau(\bar{\lambda})-2\mu|}\right) e^{-\tau(\bar{\lambda})t}, \quad \forall t \geq 0. \tag{2.70}$$

From (2.70), we see that decay rate is $\tau(\bar{\lambda})$ and taking account that $e^{-\tau(\bar{\lambda})t} < 1$, for all $t \geq 0$ we conclude (2.17). The proof of Corollary 2.1.6 is complete. □

2.4.5 State of Charge Estimation

We define a new estimator to the State of Charge by

$$\widehat{SOC}_\varphi(t) = \frac{3}{c_{\max}} \int_0^1 \widehat{c_\varphi}(t,r) r^2 \, \mathrm{d}r, \quad t \geq 0, \tag{2.71}$$

where $\widehat{c_\varphi}$ is the solution to (2.42). The following proposition gives conditions on $\widehat{SOC}_\varphi(t)$ to ensure the asymptotic convergence to the State of Charge, $SOC(t)$.

Proposition 2.4.2. *Consider φ, $L > 0$ and $\mu > 0$ satisfying Assumption 2.1.2. Let $\tau(\lambda)$ defined by (2.12). Under these assumptions there exists $\lambda_{\sup} > 2+\sqrt{6}$ such that $\tau(\lambda) > 0$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$ and there exist two cases depending on μ such that*

1. *if $\mu > \frac{\tau(\lambda)}{2}$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$, then*

$$|SOC(t) - \widehat{SOC}_\varphi(t)| \leq \frac{\sqrt{3}}{c_{\max}} \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2}{2}\frac{\lambda^3+4\lambda}{|\tau(\lambda)-2\mu|}\right)^{\frac{1}{2}} e^{-\frac{\tau(\lambda)}{2}t}, \quad \forall t \geq 0, \tag{2.72}$$

2. *if $\mu = \frac{\tau(\bar{\lambda})}{2}$ for some $\bar{\lambda} \in [2+\sqrt{6}, \lambda_{\sup}]$, then*

$$|SOC(t) - \widehat{SOC}_\varphi(t)| \leq \frac{\sqrt{3}}{c_{\max}} \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2}{2}(\bar{\lambda}^3+4\bar{\lambda})t\right)^{\frac{1}{2}} e^{-\frac{\tau(\bar{\lambda})}{2}t}, \quad \forall t \geq 0. \tag{2.73}$$

Proof. Note that

$$SOC(t) - \widehat{SOC}_\varphi(t) = \frac{3}{c_{\max}} \int_0^1 (c(t,r) - \widehat{c_\varphi}(t,r)) r^2 \,\mathrm{d}r, \quad \forall t \geq 0, \tag{2.74}$$

where $c(t,r)$ is the real concentration in the anode and $\widehat{c_\varphi}$ is the solution of (2.42). Then, using the Cauchy-Schwartz inequality on the righthand side and Theorem 2.1.3, we conclude inequalities (2.72) and (2.73) respectively and prove the statement. □

2.4.6 Proof of Theorem 2.1.7

As in the Section 2.3.3, we look for the input current $I(t)$ which allows the regulation of the $SOC(t)$ to a given reference trajectory. We use here the convergence of the estimator $\widehat{SOC}_\varphi(t)$ depending on $\varphi(t)$ instead of the surface anode concentration $c(t,1)$.

Consider the quadratic error tracking $\kappa(t) = \frac{1}{2}\left(SOC_{ref}(t) - SOC(t)\right)^2$. Then, taking the time derivative we get for all $t \geq 0$.

$$\dot{\kappa}(t) + \gamma\kappa(t) \leq \frac{\gamma}{2}\left(SOC(t) - SOC_\varphi(t)\right)^2. \tag{2.75}$$

On the other hand, from the proof of the Proposition 2.4.2 it holds for all $t \geq 0$,

$$|SOC(t) - SOC_\varphi(t)| \leq \frac{\sqrt{3}}{c_{\max}} \|\widetilde{c}(t,\cdot)\|_{L_r^2}. \tag{2.76}$$

Plugin (2.76) into (2.75), we obtain for all $t \geq 0$

$$\dot{\kappa}(t) + \gamma\kappa(t) \leq \frac{3\gamma}{2c_{\max}^2} \|\widetilde{c}(t,\cdot)\|_{L_r^2}^2. \tag{2.77}$$

Now, in virtue of Theorem 2.1.3, we have several cases.

1. Let us consider $\mu > \frac{\tau(\lambda)}{2}$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$. Then, (2.77) becomes,

$$\dot{\kappa}(t) + \gamma\kappa(t) \leq \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|_{L_r^2}^2 + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda) - 2\mu|}\right) e^{-\tau(\lambda)}, \quad t \geq 0. \tag{2.78}$$

Multiplying (2.78) by $e^{\gamma t}$, we obtain for all $t \geq 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\kappa(t)e^{\gamma t}\right) \leq \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|_{L_r^2}^2 + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda) - 2\mu|}\right) e^{(\gamma-\tau(\lambda))t} \tag{2.79}$$

Depending on γ, it holds for all $t \geq 0$ one of the followings cases:

(a) if $\tau(\lambda) < 2\mu \leq \gamma$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2(\gamma - \tau(\lambda))}\left(2\|\widetilde{c}_0\|_{L_r^2}^2 + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda) - 2\mu|}\right)\right) e^{-\tau(\lambda)t}. \tag{2.80}$$

(b) if $\tau(\lambda) < \gamma < 2\mu$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2(\gamma - \tau(\lambda))}\left(2\|\widetilde{c}_0\|_{L_r^2}^2 + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda) - 2\mu|}\right)\right) e^{-\tau(\lambda)t}. \tag{2.81}$$

(c) if $\gamma = \tau(\bar{\lambda}) < 2\mu$ for some $\bar{\lambda} \in [2+\sqrt{6}, \lambda_{\text{sup}})$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2}\left(\|\widetilde{c}_0\|_{L_r^2}^2 + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right)t\right)e^{-\gamma t}. \tag{2.82}$$

2. Let us consider $\mu = \frac{\tau(\bar{\lambda})}{2}$, for some $\bar{\lambda} \in [2+\sqrt{6}, \lambda_{sup})$. Then (2.77) becomes

$$\dot{\kappa}(t) + \gamma\kappa(t) \leq \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|_{L_r^2}^2 + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2}t\right)e^{-\tau(\bar{\lambda})t}, \quad \forall t \geq 0. \tag{2.83}$$

Multiplying by $e^{\gamma t}$ we obtain for all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\kappa e^{\gamma t}\right) \leq \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|_{L_r^2}^2 + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2}t\right)e^{(\gamma-\tau(\bar{\lambda}))t}. \tag{2.84}$$

Depending on γ, it holds for all $t \geq 0$ one of the following cases:

(a) if $2\mu = \tau(\bar{\lambda}) < \gamma$,

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2}\left(\frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2(\gamma-\tau(\bar{\lambda}))^2} + \frac{2\|\widetilde{c}_0\|_{L_r^2}^2}{(\gamma-\tau(\bar{\lambda}))} + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2(\gamma-\tau(\bar{\lambda}))}t\right)\right)e^{-\tau(\bar{\lambda})t}. \tag{2.85}$$

(b) if $2\mu = \tau(\bar{\lambda}) = \gamma$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|_{L_r^2}^2 t + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{4}t^2\right)\right)e^{-\gamma t}, \ \forall t \geq 0. \tag{2.86}$$

(c) if $\gamma < 2\mu = \tau(\bar{\lambda})$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2}\left(\frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2(\gamma-\tau(\bar{\lambda}))^2} + \frac{2\|\widetilde{c}_0\|_{L_r^2}^2}{|\gamma-\tau(\bar{\lambda})|} + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2|\gamma-\tau(\bar{\lambda})|}\right)\right)e^{-\gamma t}. \tag{2.87}$$

The proof of Theorem 2.1.7 is complete. □

Remark 2.4.3. *We have proved that the input current $I(t)$ given by (2.18), forces the system (2.7) to track the signal SOC_{ref} with an exponential decay rate.*

Following ideas of the proof of the Corollaries 2.1.5 and 2.1.6, see Sections 2.4.3 and 2.4.4 respectively, we would get similar results as Corollaries 2.1.5 and 2.1.6 in the context of Theorem 2.1.7.

2.5 Simulations

In this section we present some simulations to illustrate Theorem 2.3.8 and Theorem 2.1.7. The model parameters used in this work have been taken from the online repository [63] (please also see the related paper [64]). We perform some simulations in two cases. Section 2.5.1 uses boundary measurements as output while the case where we dispose of the estimator φ is simulated in Section 2.5.2.

In both types of simulations the definition of $p_0(\lambda)$, $p_1(r,\lambda)$, and $\widehat{SOC}(t)$ are given by (2.10), (2.11) and (2.34), respectively. Moreover, we set the initial conditions of system in closed loop with an error of 50% with respect to the original value. The values for the remain parameters are shown in Table 2.2.

Parameters	Values
$c(0,r)$	$1.5c_0$
$\widehat{c}(0,r)$	$1.5c_0$
$c_{\max}$	$2.5 \cdot 10^4$
λ	5
γ	70

TABLE 2.2: Parameter simulations

Concerning discretization, we have used central difference method in the spatial variable to get the corresponding ODE system, which is solved with the MatLab routine *ode23tb*. Notice that the system (2.7) has a singularity at $r = 0$. Therefore, in order to obtain the corresponding ODE, we have done the following approximation. Consider the limit

$$c_t(t,0) = \lim_{r\to 0}\left(\frac{2}{r}c_r(t,r) + c_{rr}(t,r)\right). \tag{2.88}$$

Then, by the L'Hôpital's rule and the boundary condition at $r = 0$ of (2.7) we get that

$$\lim_{r\to 0}\frac{c_r(t,r)}{r} = c_{rr}(t,0).$$

Thus, we have that

$$c_t(t,0) \approx 3c_{rr}(t,0). \tag{2.89}$$

In a similar way, we get the following approximation at $r = 0$

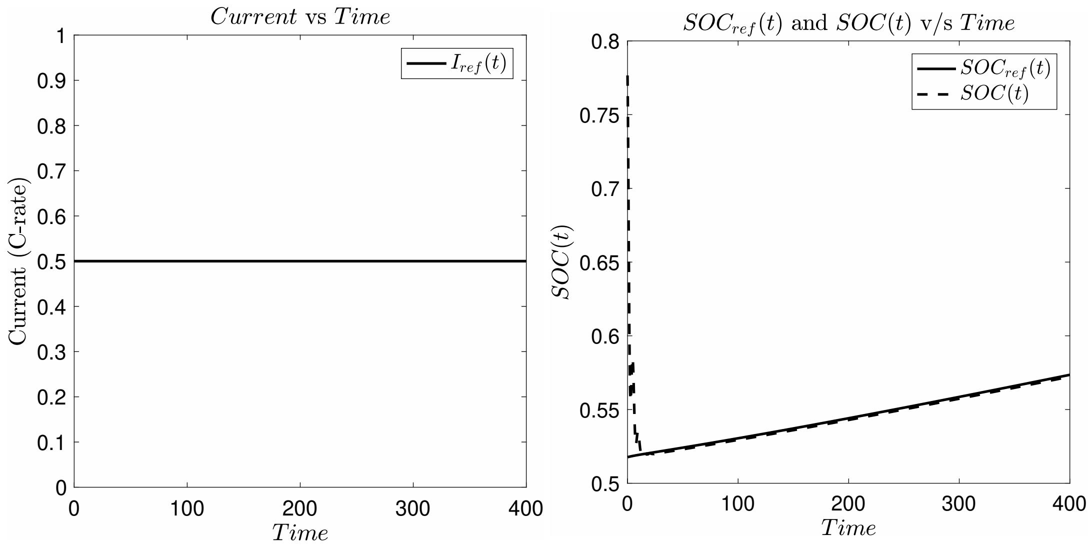
$$\widehat{c}_t(t,0) \approx 3\widehat{c}_{rr}(t,0) + p_1(0)(c(t,1) - \widehat{c}(t,1)). \tag{2.90}$$

2.5.1 Tracking using output

First, we generate numerical data for the illustration of Theorem 2.3.8. We set a reference current input $I_{ref}(t)$ and constant initial condition $c_0 = 1.2901 \cdot 10^4$. We simulate to obtain a SOC signal which is used as the reference $SOC_{ref}(t)$ in our simulations. Then, we simulate the closed-loop system (2.7), (2.22) and (2.38), which is

$$\begin{cases} c_t(t,r) = \frac{2}{r}c_r(t,r) + c_{rr}(t,r), \\ c_r(t,0) = 0,\ c_r(t,1) = \dot{SOC}_{ref}(t) + \gamma\left(SOC_{ref}(t) - \frac{3}{c_{\max}}\int_0^1 \widehat{c}(t,r)r^2dr\right), \\ c(0,r) = c_0(r), \\ \widehat{c}_t(t,r) = \frac{2}{r}\widehat{c}_r + \widehat{c}_{rr} + p_1(r,\lambda)(c(t,1) - \widehat{c}(t,1)), \\ \widehat{c}_r(t,0) = 0, \\ \widehat{c}_r(t,1) = \dot{SOC}_{ref}(t) + \gamma\left(SOC_{ref}(t) - \frac{3}{c_{\max}}\int_0^1 \widehat{c}(t,r)r^2dr\right) \\ +p_0(\lambda)(c(t,1) - \widehat{c}(t,1)), \\ \widehat{c}(0,r) = \widehat{c}_0(r). \end{cases} \tag{2.91}$$

We have done the previous strategy for two different situations. First we take a constant signal $I_{ref}(t) = 0.5C$ as the input current used to generate the state of charge reference $SOC_{ref}(t)$. We see in Figure 2.2 the input $I_{ref}(t) = 0.5C$ (on the left) and a good performance of the $SOC(t)$ tracking trajectory (on the right). Then we do the same simulations in the case of a square signal $I_{ref}(t) = 4.5square(\frac{64}{900\pi}t)C$. The results can be seen in Figure 2.3. These simulations illustrate an exponential rate for the tracking.



(a) The input $I_{ref}(t) = 0.5C$

(b) $SOC(t)$ and $SOC_{ref}(t)$

FIGURE 2.2: (*Left*) The input $I_{ref}(t) = 0.5C$. (*Right*) We compare $SOC(t)$ for the controlled system with the reference $SOC_{ref}(t)$, generated by $I_{ref}(t)$.

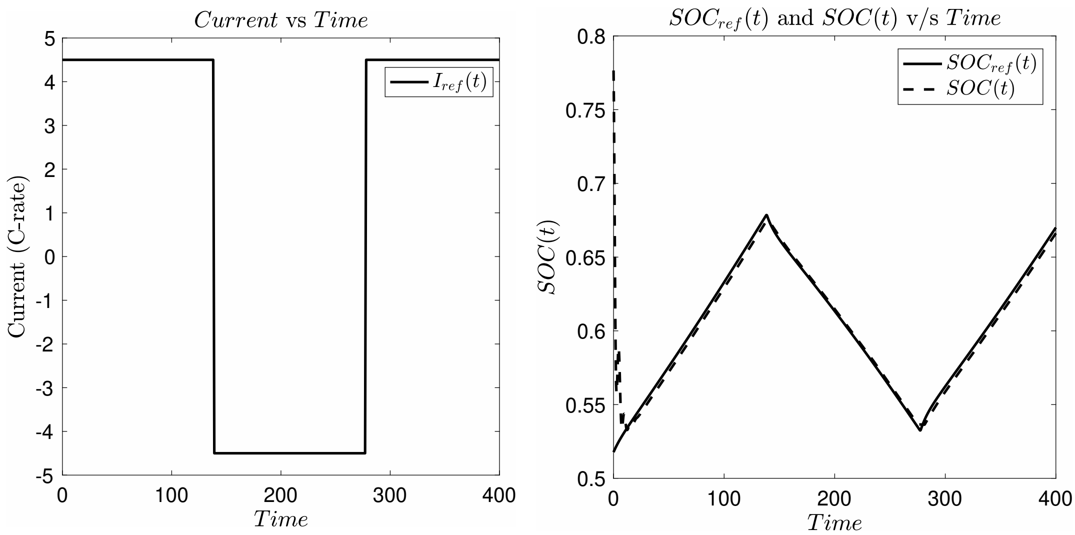
2.5.2 Tracking using output estimator

As above, we generate a synthetic state of charge $SOC_{ref}(t)$ from a known $I_{ref}(t)$ and then we simulate the controlled system (2.7), (2.42) and (2.18) that is

$$\begin{cases} c_t(t,r) = \frac{2}{r}c_r(t,r) + c_{rr}(t,r), \\ c_r(t,0) = 0,\ c_r(t,1) = S\dot{O}C_{ref}(t) + \gamma\left(SOC_{ref}(t) - \frac{3}{c_{\max}}\int_0^1 \widehat{c}_\varphi(t,r)r^2dr\right), \\ c(0,r) = c_0(r), \\ \partial_t\widehat{c}_\varphi(t,r) = \frac{2}{r}\partial_r\widehat{c}_\varphi + \partial_{rr}\widehat{c}_\varphi + p_1(r,\lambda)(\varphi(t) - \widehat{c}_\varphi(t,1)), \\ \widehat{c}_\varphi(t,0) = 0,\ \partial_r\widehat{c}_\varphi(t,1) = S\dot{O}C_{ref}(t) + \gamma\left(SOC_{ref}(t) - \frac{3}{c_{\max}}\int_0^1 \widehat{c}_\varphi(t,r)r^2dr\right) \\ +p_0(\lambda)(\varphi(t) - \widehat{c}_\varphi(t,1)), \\ \widehat{c}_\varphi(0,r) = \widehat{c}_{\varphi 0}(r). \end{cases} \tag{2.92}$$

Note that instead of $c(t,1)$, in this simulation, we have used an artificial estimator $\varphi(t)$ of the boundary concentration $c(t,1)$, namely $\varphi(t) = c(t,1) + Me^{-\mu t}$, with $M > 0$ and $\mu > 0$. This $\varphi(t)$ satisfies the Assumption 2.1.2. To run out the simulations we have used the parameters values given by the Table 2.2 and set up $M = c_{\max}$ and $\mu = 70$ to characterize the estimator $\varphi(t)$.

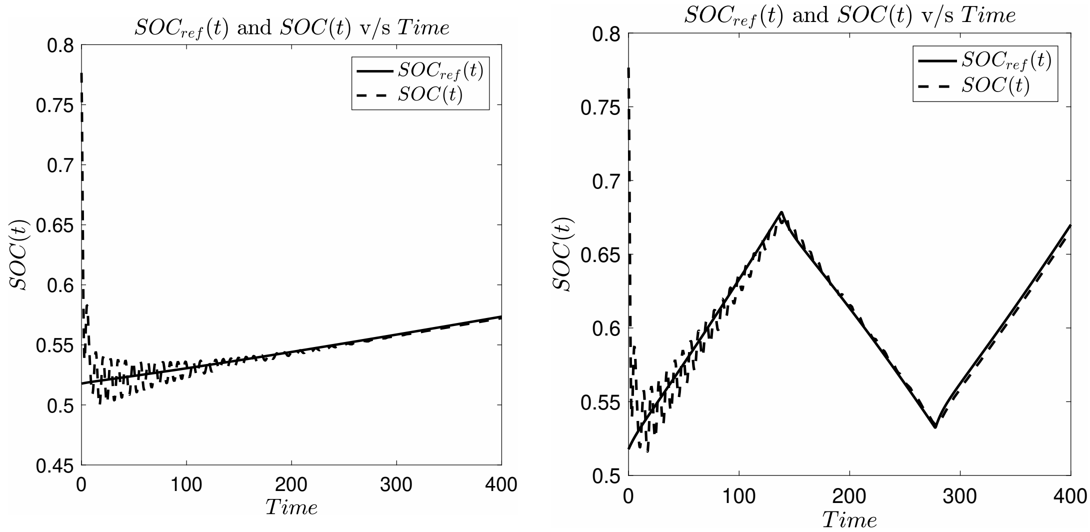
As in Section 2.5.1 we run simulations in two cases. First for $I_{ref}(t) = 0.5C$ and then for $I_{ref}(t) = 4.5square(\frac{64}{900\pi}t)C$. The results of the tracking of $SOC(t)$ to the reference $SOC_{ref}(t)$ are presented in Figure 2.4 confirming the good performance of



(a) The input $I_{ref}(t) = 4.5\, square(\frac{64}{900\pi}t)C$ (b) $SOC(t)$ and $SOC_{ref}(t)$

FIGURE 2.3: ($Left$) The input $I_{ref}(t) = 4.5\, square(\frac{64}{900\pi}t)C$. ($Right$) We compare $SOC(t)$ for the controlled system with the reference $SOC_{ref}(t)$, generated by $I_{ref}(t)$.

our controllers. As predicted by Theorem 2.1.7, in simulations the convergence seems to be of exponential type.



(a) Input $I_{ref}(t) = 0.5C$ (b) $SOC(t)$ and $SOC_ref(t)$

FIGURE 2.4: We compare $SOC(t)$ for the controlled system with the reference $SOC_{ref}(t)$ generated by $I_{ref}(t) = 0.5C$ (Left) and $I_{ref}(t) = 4.5square(\frac{64}{900\pi}t)C$ (Right).

Chapter 3

Null controllability of some parabolic-elliptic systems

This chapter is submitted for publication in

E. Hernández, C. Prieur, and E. Cerpa. Boundary null controllability of some parabolic-elliptic systems. *Submitted*, 2021.

Summary

3.1 Introduction

Our work is motivated by the many relevant and wide-variety of the applications involving weakly or strongly coupled parabolic-elliptic systems with different boundary conditions. For example, in biology, the Keller-Segel system is used to describe the chemotaxis phenomena, [35]. Besides, parabolic-elliptic systems also arise in the study of the groundwater problem, that is the description of the movement of a fluid of variable density thought a porous medium under the influence of gravity and hydrodynamic dispersion. See for instance [13]. Parabolic-elliptic systems also arise in semiconductor modeling, see for example [61, 45]. In [27], the authors mention that, parabolic-elliptic systems are used to describe the interaction of two scalar populations, when the time scale of growth rate of one population is much greater than the other one.

Concerning to the literature involving control problems, we can mention that, there exist several recent results about the null controllability of a parabolic-elliptic systems under the action of a control locally distributed, see for instance [72, 26, 27]. In the same direction, we could mention [87] as well, where the authors prove the approximate controllability of the Camassa-Holm equation by the action of a bilinear lumped control, which is a time-dependent function and appears as a coefficient in the main equation, and a locally distributed control. In [21], the authors use an optimal control framework to solve the inverse problem of identifying the diffusion coefficient in a coupled parabolic-elliptic system. In [30], the authors study the null controllability of a family of equations called *b-equations*, that can be viewed as an asymptotically equivalent approximation of the shallow water equations. By a change of variables these equations can be re-written as a parabolic-elliptic system. The authors prove the null controllability of the system by the action of distributed control acting on the parabolic part. Moreover, in [30] the authors provide a boundary control result for the cases when a control is acting on the boundary of the parabolic and elliptic part at the same time.

The purpose of this paper is contributing to the study of controllability properties of parabolic-elliptic systems controlled by a one single scalar control, acting only on one boundary corresponding to the parabolic or elliptic part of the equation.

In a first case, we consider the following control system

$$\begin{cases} z_t - z_{xx} + qz = f(z) + \zeta, & (t,x) \in (0,T) \times (0,L), \\ -\zeta_{xx} + \gamma\zeta = z & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = u(t), \quad z(t,L) = 0, & t \in (0,T), \\ \zeta(t,0) = 0, \quad \zeta(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{3.1}$$

where $T > 0$, $L > 0$, the state is given by (z, ζ), γ, $q \in L^\infty(0,L)$, $f \in W^{2,\infty}(\mathbb{R})$ is a nonlinear function and the time-dependent function u is a boundary control acting on the parabolic boundary condition.

In a second case, we consider the system given by

$$\begin{cases} z_t - z_{xx} + q_0 z = \zeta, & (t,x) \in (0,T) \times (0,L), \\ -\zeta_{xx} + \gamma_0\zeta = z & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = 0, \quad z(t,L) = 0, & t \in (0,T), \\ \zeta(t,0) = u(t), \quad \zeta(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{3.2}$$

where $T > 0$, $L > 0$, the state is given by (z, ζ), γ_0, q_0 are scalar constants, and the time-dependent function u is a boundary control acting on the boundary of the elliptic equation.

3.1.1 Problem statement and main results

For the systems (3.1) and (3.2), we are interested in studying the null controllability by the action of a one single control placed at the boundary. In order to be more precise, we are interested in knowing if the system (3.1) or (3.2) possess the following control property. Given $T > 0$ and appropriate space X, we say that system (3.1) or (3.2) is null controllable if for any initial condition $z_0 \in X$, there exists a boundary control u such that the solution to (3.1) or (3.2) with $z(0,\cdot) = z_0$ satisfies $(z(T,\cdot), \zeta(T,\cdot)) = (0,0)$.

To study the systems (3.1) and (3.2) let us set the following operator

$$F_\gamma : g \in L^2(0,L) \longmapsto F_\gamma(g) = \zeta \in H_0^1(0,L), \tag{3.3}$$

where ζ is the solution to

$$\begin{cases} -\zeta_{xx} + \gamma\zeta = g, \quad x \in (0,L), \\ \zeta(0) = 0, \zeta(L) = 0, \end{cases} \tag{3.4}$$

with $\gamma \in L^\infty(0,L)$. We collect some properties of the operator F_γ in the following lemma, whose proof is given in the Appendix (Section A.5).

Lemma 3.1.1. *Let $g \in L^2(0,L)$ and $\gamma \in L^\infty(0,L)$ such that $\gamma(x) \geq \gamma_0 > -\pi^2/L^2$ for all $x \in [0,L]$. Then, the operator F_γ defined by (3.3) is well defined, linear continuous* *and self-adjoint. Moreover, there exists a positive constant $C = C(\gamma_0, L)$ such that*

$$\|F_\gamma(g)\|_{H_0^1(0,L)} \leq C(\gamma_0, L)\|g\|_{L^2(0,L)}, \quad \forall g \in L^2(0,L), \tag{3.5}$$

where $C(\gamma_0, L) = \max\left\{\frac{L}{\pi}, \frac{L\pi}{\pi^2+\gamma_0 L^2}\right\}$.

As a consequence of Lemma 3.1.1 we are able to re-write the system (3.1) in a equivalent way as follows

$$\begin{cases} z_t - z_{xx} + q(x)z = f(z) + F_\gamma(z), & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = u(t), \quad z(t,L) = 0, & t \in (0,T), \\ z(x,0) = z_0(x), & x \in (0,L). \end{cases} \tag{3.6}$$

To prove the null controllability of the system (3.6), as usual in this kind of problems, we begin by proving the boundary null controllability of the following linear system

$$\begin{cases} z_t - z_{xx} + (q - f'(0))z - F_\gamma(z) = g(t,x), & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = u(t), \quad z(L,t) = 0, & t \in (0,T), \\ z(x,0) = z_0(x), & x \in (0,L), \end{cases} \tag{3.7}$$

where F_γ is the operator given by (3.3). Notice that if $g = 0$ we recover the linearized system of (3.6) around $z = 0$.

At this point, it is worth to mention that unlikely the case of the linear heat equation, here the null controllability with internal control does not imply the null controllability with a boundary control. This is due to the coupling with the elliptic equation. Thus, the argument of enlarging the interval, putting a distributed control outside of original domain and then taking the trace of the solution, fails if we do not increase the number of the controls, see for example Section 5.2 in [30]. This leads to us to deal with this problem in an independent way from the distributed control problem.

In order to prove the boundary null controllability of the linear system (3.7), see Proposition 3.3.4, we use the controllability-observability duality principle. To do that, we use the Carleman estimate with boundary observation to deduce an observability inequality for the adjoint system to (3.7). Then, we show that the local boundary null controllability property holds for the nonlinear control system (3.6) by using a local inverse function argument. The first main result of this chapter can be summarized as follows.

Theorem 3.1.2. *Let $T > 0$, $L > 0$, $\gamma, q \in L^\infty(0,L)$, $f \in W^{2,\infty}(\mathbb{R})$ such that $\gamma(x) \geq \gamma_0 > -\pi^2/L^2$, for all $x \in [0,L]$ and $q(x) \geq q_0$ such that $q_0 + f'(0) \geq C(\gamma_0, L)L/\pi - (\pi/L)^2$, for all $x \in [0,L]$. Then, the system (3.6) is locally null controllable. That is, there exists $r > 0$ such that for any $z_0 \in H^{-1}(0,L)$ such that $\|z_0\|_{H^{-1}(0,L)} \leq r$, there exists $u \in L^2(0,T)$ and $z \in C([0,T]; H^{-1}(0,L)) \cap L^2(0,T; L^2(0,L))$ solution to (3.6). Moreover it holds $z(T,x) = 0$.*

Now, in order to prove the boundary null controllability for the system (3.2) we introduce a lift function $\xi \in C^2([0,L])$, such that $\xi(0) = 1$ and $\xi(L) = 0$ and let define the following change of variable

$$\tilde{\zeta}(t,x) = \zeta(t,x) - \xi(x)u(t). \tag{3.8}$$

Then, the system (3.2) can be rewritten as follows

$$\begin{cases} z_t - z_{xx} + q_0 z = \tilde{\zeta} + \xi u, & (t,x) \in (0,T) \times (0,L), \\ -\tilde{\zeta}_{xx} + \gamma_0 \tilde{\zeta} = z - (-\xi_{xx} + \gamma_0 \xi)\, u & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = 0, \quad z(t,L) = 0, & t \in (0,T), \\ \tilde{\zeta}(t,0) = 0, \quad \tilde{\zeta}(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{3.9}$$

Now, we use the operator F_γ defined by (3.3), in order to reduce the system as before. Note that, by linearity, $F_{\gamma_0}(z - (-\xi_{xx} + \gamma_0 \xi)) = F_{\gamma_0}(z) - uF_{\gamma_0}(-\xi_{xx} + \gamma_0 \xi) = \zeta$. Thus, we see that system (3.2) is equivalent to the following system

$$\begin{cases} z_t - z_{xx} + q_0 z - F_{\gamma_0}(z) = \theta u, & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = 0, \quad z(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{3.10}$$

where $\theta = \xi - F_{\gamma_0}(-\xi_{xx} + \gamma_0 \xi)$. Now, we summarize the second main result of this chapter.

Theorem 3.1.3. *Let $T > 0$, $L > 0$, and constant coefficients q_0 and γ_0 such that $\gamma_0 > -\pi^2/L^2$, and q_0 such that $q_0 \geq C(\gamma_0, L)L/\pi - (\pi/L)^2$. Then system (3.10) is null controllable. That is, for any $z_0 \in L^2(0,L)$ there exists $u \in L^2(0,T)$ and $z \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^1(0,L))$ solution to (3.10). Moreover it holds $z(T,\cdot) = 0$.*

To prove Theorem 3.1.3 we use a problem of moments approach and the spectral analysis of the underlying spatial operator

$$\begin{aligned} A_0 : D(A_0) \subset L^2(0,L) &\longrightarrow L^2(0,L), \\ A_0 v &\longmapsto v_{xx} - q_0 v + F_{\gamma_0}(v). \end{aligned} \tag{3.11}$$

3.1.2 Organization

This chapter is organized as follows. First, in Section 3.2 we stablish the well-posedness framework for the systems. Next, Section 3.3 is devoted to the proof of the control results presented in this introduction section. More precisely, the Section 3.3.1 is dedicated to the proof of Proposition 3.3.4, which state the null controllability of the linear system (3.7) and in Section 3.3.2 we give the proof of the Theorem 3.1.2. In Section 3.3.3 the proof of the Theorem 3.1.3 is given.

3.2 Well-posedness and regularity of solutions

Along this section, we prove some well-posedness results in order to get a suitable functional space framework for the control systems (3.6) and (3.10).

3.2.1 Well-posedness of the control system (3.6)

In order to get a proper framework for the nonlinear control system (3.6), we begin by a stating a well-posedness framework for the linearized system (3.7) and then, by means of a fix point argument, we state a local well-posedness result for the control system (3.6).

Linearized system (3.7)

Consider the following operator

$$A : \phi \in D(A) \subset L^2(0,L) \longmapsto \phi_{xx} - (q - f'(0))\phi + F_\gamma(\phi) \ \in L^2(0,L), \tag{3.12}$$

with domain $D(A) := H^2(0,L) \cap H_0^1(0,L)$. The operator A is self-adjoint. Indeed, $(A\phi_1, \phi_2)_{L^2(0,L)} = (\phi_1, A\phi_2)_{L^2(0,L)}$, for all $w_1, w_2 \in D(A)$, and $D(A^*) = H^2(0,L) \cap H_0^1(0,L)$.

Since that $\|F_\gamma(\phi)\|_{L^2(0,L)} \leq \frac{L}{\pi}\|F_\gamma(\phi)\|_{H_0^1(0,L)}$, it is not difficult to see that A also satisfies, for all $\phi \in H^2(0,L) \cap H_0^1(0,L)$

$$(A\phi, \phi)_{L^2(0,L)} \leq \left(-\frac{\pi^2}{L^2} - q_0 - f'(0) + \frac{L}{\pi}C(\gamma_0, L)\right)\|\phi\|^2_{L^2(0,L)},$$

where we used that $q(x) \geq q_0$, for all $x \in [0,L]$. Above inequality lead to us to impose that

$$q_0 + f'(0) > \frac{L}{\pi}C(\gamma_0, L) - \frac{\pi^2}{L^2}. \tag{3.13}$$

Then it follows that

$$(A\phi, \phi)_{L^2(0,L)} \leq 0, \quad \forall \phi \in H^2(0,L) \cap H_0^1(0,L), \tag{3.14}$$

in other words, A is dissipative. Moreover, A is a m-dissipative operator (e.g Corollary 2.4.8 in [9]) and by the Hille-Yosida-Phillips Theorem, see for instance [9, Theorem 3.4.4], we conclude that A is a generator of a contraction semigroup in $L^2(0,L)$. Then, if $z_0 \in D(A)$, $g \in C^1([0,T]; L^2(0,L))$ and $u(t) = 0$, the solution to (3.7) satisfies $z \in C([0,T]; D(A)) \cap C^1([0,T]; L^2(0,L))$, see [9, Proposition 4.1.6].

We need more precise information about the regularity of the solution, let us perform some energy estimations in order to get it.

Proposition 3.2.1. *Consider G being either $L^2(0,T;L^2(0,L))$ or $L^1(0,T;H_0^1(0,L))$. Let $g \in G$, $u = 0$ and $\gamma(x) \geq \gamma_0 > -\pi^2/L^2$, for all $x \in [0,L]$ and $q(x) \geq q_0$ such that $q_0 + f'(0) \geq C(L,\gamma_0)L/\pi - (\pi/L)^2$, for all $x \in [0,L]$ in (3.7). If $z_0 \in H_0^1(0,L)$, then (3.7) has a unique solution $z \in C([0,T]; H_0^1(0,L)) \cap L^2(0,T; H^2(0,L) \cap H_0^1(0,L))$. Moreover, there exists $C > 0$ such that*

$$\|z\|_{L^\infty(0,T;H_0^1(0,L)) \cap L^2(0,T;H^2(0,L)\cap H_0^1(0,L))} \leq C(\|g\|_G + \|z_0\|_{H_0^1(0,L)}). \tag{3.15}$$

Proof. Let $t \in [0,T]$ and multiplying equation (3.7) by z_{xx}, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z_x^2 \,\mathrm{d}x + \int_0^L z_{xx}^2 \,\mathrm{d}x = -\int_0^L g z_{xx} \,\mathrm{d}x + \int_0^L (q - f'(0)) z z_{xx} \,\mathrm{d}x - \int_0^L F_\gamma(z) z_{xx} \,\mathrm{d}x. \tag{3.16}$$

Let us assume that $g \in L^2(0,T,L^2(0,L))$. Using the Cauchy-Schwartz and Young's inequalities and the continuity of the operator F_γ, see Lemma 3.1.1, on the right-hand side of (3.16) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z_x^2\,\mathrm{d}x + \int_0^L z_{xx}^2\,\mathrm{d}x \leq \frac{3}{2}\int_0^L g^2\,\mathrm{d}x + \frac{1}{6}\int_0^L z_{xx}^2\,\mathrm{d}x + \frac{3\|q-f'(0)\|^2_{L^\infty(0,L)}}{2}\int_0^L z^2\,\mathrm{d}x$$
$$+\frac{1}{6}\int_0^L z_{xx}^2\,\mathrm{d}x + \frac{3L^2C^2(\gamma_0,L)}{2\pi^2}\int_0^L z^2\,\mathrm{d}x + \frac{1}{6}\int_0^L z_{xx}^2\,\mathrm{d}x. \quad (3.17)$$

Then rearranging terms and using the Poincaré inequality on the right-hand side of (3.17), it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z_x^2\,\mathrm{d}x + \int_0^L z_{xx}^2\,\mathrm{d}x \leq 3\int_0^L g^2\,\mathrm{d}x$$
$$+\frac{3L^2}{\pi^2}\left(\|q-f'(0)\|^2_{L^\infty(0,L)} + \frac{L^2}{\pi^2}C^2(\gamma_0,L)\right)\int_0^L z_x^2\,\mathrm{d}x. \quad (3.18)$$

Since that $\int_0^L z_{xx}^2\,\mathrm{d}x \geq 0$ for all $t\in[0,T]$, the left-hand side of (3.18) is lower bounded by $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z_x^2\,\mathrm{d}x$. Then, integrating (3.18) in $(0,t)$ we obtain

$$\int_0^L z_x^2(t,x)\,\mathrm{d}x \leq 3\int_0^t\int_0^L g^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^L z_{0x}^2\,\mathrm{d}x$$
$$+\frac{3L^2}{\pi^2}\left(\|q-f'(0)\|^2_{L^\infty(0,L)} + \frac{L^2}{\pi^2}C^2(\gamma_0,L)\right)\int_0^t\int_0^L z_x^2\,\mathrm{d}x\,\mathrm{d}s. \quad (3.19)$$

It follows from the Grönwall Lemma that, for all $t\in[0,T]$

$$\int_0^L z_x^2(t,x)\,\mathrm{d}x \leq$$
$$\exp\left\{\frac{3L^2}{\pi^2}\left(\|q-f'(0)\|^2_{L^\infty(0,L)} + \frac{L^2}{\pi^2}C^2(\gamma_0,L)\right)T\right\}\left(3\int_0^T\int_0^L g^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^L z_{0x}^2\,\mathrm{d}x\right). \quad (3.20)$$

Then, plug-in (3.20) into (3.18) and integrating in $[0,T]$, we can conclude that there exists a constant $C>0$ such that

$$\|z\|_{L^\infty(0,T;H_0^1(0,L))\cap L^2(0,T;H^2(0,L)\cap H_0^1(0,L))} \leq$$
$$C\left(\|g\|_{L^2(0,T;L^2(0,L))} + \|z_0\|_{H_0^1(0,L)}\right). \quad (3.21)$$

Thus, by a density argument we can prove that if $g\in L^2(0,T;L^2(0,L))$ and $z_0\in H_0^1(0,L)$, the solution z belongs to $C([0,T];H_0^1(0,L))\cap L^2(0,T;H^2(0,L)\cap H_0^1(0,L))$.

Now, let us consider $g \in L^1(0,T;H_0^1(0,L))$. As before multiplying equation (3.7) by z_{xx}, then we perform integrations by parts in $[0,L]$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z_x^2\,\mathrm{d}x + \int_0^L z_{xx}^2\,\mathrm{d}x = \int_0^L g_x z_x\,\mathrm{d}x + \int_0^L (q-f'(0))zz_{xx}\,\mathrm{d}x - \int_0^L F_\gamma(z)z_{xx}\,\mathrm{d}x. \tag{3.22}$$

Using the Cauchy-Schwartz inequality, Young's inequality and continuity of F_γ on the right-hand side of (3.22) we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z_x^2\,\mathrm{d}x + \int_0^L z_{xx}^2\,\mathrm{d}x \le \|g_x\|_{L^2(0,L)}\|z_x\|_{L^2(0,L)} + \|q-f'(0)\|^2_{L^\infty(0,L)}\int_0^L z^2\,\mathrm{d}x \\ + \frac{1}{4}\int_0^L z_{xx}^2\,\mathrm{d}x + \int_0^L F_\gamma^2(z)\,\mathrm{d}x + \frac{1}{4}\int_0^L z_{xx}^2\,\mathrm{d}x.\end{aligned} \tag{3.23}$$

It follows from the Poincaré inequality and the continuity of the operator F_γ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z_x^2\,\mathrm{d}x + \int_0^L z_{xx}^2\,\mathrm{d}x \le 2\|g_x\|_{L^2(0,L)}\|z_x\|_{L^2(0,L)} \\ + \frac{2L^2}{\pi^2}\left(\|q-f'(0)\|^2_{L^\infty(0,L)} + \frac{L^2}{\pi^2}C^2(\gamma_0,L)\right)\int_0^L z_x^2\,\mathrm{d}x.\end{aligned} \tag{3.24}$$

Now, since that $\int_0^L z_{xx}^2\,\mathrm{d}x \ge 0$ we ignore it and integrating in $(0,t)$ we obtain

$$\begin{aligned}\int_0^L z_x^2\,\mathrm{d}x \le 2\int_0^t \|g_x\|_{L^2(0,L)}\|z_x\|_{L^2(0,L)}\,\mathrm{d}t + \|z_{0x}\|^2_{L^2(0,L)} \\ + \frac{2L^2}{\pi^2}\left(\|q-f'(0)\|^2_{L^\infty(0,L)} + \frac{L^2}{\pi^2}C^2(\gamma_0,L)\right)\int_0^t\int_0^L z_x^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.25}$$

Then, by the Grönwall Lemma, there exists a positive constant C_0, such that for all $t \in [0,T]$

$$\int_0^L z_x^2\,\mathrm{d}x \le C_0\left(2\int_0^T \|g_x\|_{L^2(0,L)}\|z_x\|_{L^2(0,L)}\,\mathrm{d}t + \|z_{0x}\|^2_{L^2(0,L)}\right). \tag{3.26}$$

where, $C_0 = e^{\left\{\frac{2L^2}{\pi^2}\left(\|q-f'(0)\|^2_{L^\infty(0,L)} + \frac{L^2}{\pi^2}C^2(\gamma_0,L)\right)T\right\}}$.

Note that, by the Holder inequality it holds,

$$\int_0^T \|g_x\|_{L^2(0,L)}\|z_x\|_{L^2(0,L)}\,\mathrm{d}t \le \|g_x\|_{L^1(0,T;L^2(0,L))}\|z_x\|_{L^\infty(0,T;L^2(0,L))} \tag{3.27}$$

and by the Young's inequality we get, for all $\alpha > 0$ that

$$2\|g_x\|_{L^1(0,T;L^2(0,L))}\|z_x\|_{L^\infty(0,T;L^2(0,L))} \leq \alpha\|g_x\|^2_{L^1(0,T;L^2(0,L))} + \frac{1}{\alpha}\|z_x\|^2_{L^\infty(0,T;L^2(0,L))}. \tag{3.28}$$

Combining (3.26), and a suitable α in (3.28), we obtain that there exists $C_1 > 0$ such that for all $t \in [0,T]$, it holds

$$\|z_x(t,\cdot)\|_{L^2(0,L)} \leq C_1\left(\|g_x\|_{L^1(0,T;L^2(0,L))} + \|z_{0x}\|_{L^2(0,L)}\right) + \frac{1}{2}\|z_x\|_{L^\infty(0,T;L^2(0,L))}. \tag{3.29}$$

Reducing terms we get

$$\|z_x\|_{L^\infty(0,T;L^2(0,L))} \leq 2C_1\left(\|g_x\|_{L^1(0,T;L^2(0,L))} + \|z_{0x}\|_{L^2(0,L)}\right). \tag{3.30}$$

Now, we integrate (3.24) in [0,T] and then we plug-in (3.30). By the continuous injection of $L^\infty(0,T)$ into $L^2(0,T)$ there exists $C_2 > 0$ such that

$$\|z\|_{L^\infty(0,T;H_0^1(0,L))\cap L^2(0,T;H^2(0,L)\cap H_0^1(0,L))} \leq C_2\left(\|g\|_{L^1(0,T;H_0^1(0,L))} + \|z_0\|_{H_0^1(0,L)}\right). \tag{3.31}$$

Finally, using a density argument we complete the proof of Proposition 3.2.1. □

In order to be able to define the solution of (3.7) with a non homogeneous boundary condition $u \in L^2(0,T)$, we need the following corollary.

Corollary 3.2.2. *Under the assumptions of Proposition 3.2.1. There exists $C > 0$ such that the solution z of (3.7) satisfies*

$$\|z_x(t,0)\|_{L^2(0,T)} \leq C(\|g\|_G + \|z_0\|_{H_0^1(0,L)}) \tag{3.32}$$

Proof. This inequality is a direct consequence of Proposition 3.2.1 and the continuous injection of $H^2(0,L)$ into $C^1([0,L])$. □

In order to give sense to the solution of (3.7) in a less regular framework, that is, with data $u(t) \in L^2(0,T)$ and $z_0 \in H^{-1}(0,L)$, let us considerate the next formal computations. Consider (3.7) and multiply by w, then perform some integration by parts, we get

$$\begin{aligned}&\int_0^T\int_0^L z(t,x)\left(-w_t(t,x) - w_{xx}(t,x) + (q - f'(0))w(t,x) - F_\gamma(w)\right)\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^L zw|_0^T\,\mathrm{d}x + \int_0^T z_x w|_0^L\,\mathrm{d}t + \int_0^T zw_x|_0^L\,\mathrm{d}t = \int_0^T\int_0^L g(t,x)w(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.33}$$

Let us consider $h \in L^2(0,T;L^2(0,L))$ or $h \in L^1(0,T;H_0^1(0,L))$ and w solution to the following equation

$$\begin{cases} -w_t - w_{xx} + (q - f'(0))w - F_\gamma(w) = h, & (t,x) \in (0,T)\times(0,L),\\ w(t,0) = 0, \quad w(L,t) = 0, & t \in (0,T),\\ w(T,x) = 0, & x \in (0,L). \end{cases} \tag{3.34}$$

The above equation is well-posed. Indeed, thanks to the change of variable $t \to T-t$ and the Proposition 3.2.1, the equation (3.34) has a unique solution $w \in C([0,T];H_0^1(0,L)) \cap L^2(0,T;H^2(0,L)\cap H_0^1(0,L))$.

Now, plugging (3.34) into (3.33), we obtain

$$\int_0^T \int_0^L z(t,x)h(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^L z_0(x)w(x,0)\,\mathrm{d}x + \int_0^T u(t)w_x(t,0)\,\mathrm{d}t + \int_0^T \int_0^L gw\,\mathrm{d}x\,\mathrm{d}t \tag{3.35}$$

In order to give sense to the previous formal computations, we present the following definition

Definition 3.2.3. *Let $z_0 \in H^{-1}(0,L)$, $g \in L^1(0,T;H^{-1}(0,L))$, $u \in L^2(0,T)$ and γ, $q \in L^\infty(0,L)$, $\gamma(x) \geq \gamma_0 > -\pi^2/L^2$, for all $x \in [0,L]$ and $q(x) \geq q_0$ such that $q_0 + f'(0) \geq C(L,\gamma_0)L/\pi - (\pi/L)^2$, for all $x \in [0,L]$. A solution of (3.7) defined by transposition is a function $z \in L^2(0,T;L^2(0,L))$ such that for any $h \in L^2(0,T;L^2(0,L))$,*

$$\begin{aligned}\int_0^T \int_0^L z(t,x)h(t,x)\,dx\,dt &= \langle z_0, w(0,x)\rangle_{H^{-1}(0,L),H_0^1(0,L)} \\ &\quad + \langle g, w\rangle_{L^1(0,T;H^{-1}(0,L)),L^\infty(0,T;H_0^1(0,L))} + \int_0^T u(t)w_x(t,0)\,dt,\end{aligned} \tag{3.36}$$

where w is the unique solution to (3.34).

The following proposition establish the existence and uniqueness of the solutions to (3.7) defined by transposition.

Proposition 3.2.4. *Let $z_0 \in H^{-1}(0,L)$, $g \in L^1(0,T;H^{-1}(0,L))$, $u \in L^2(0,T)$ and γ, $q \in L^\infty(0,L)$ such that $\gamma(x) \geq \gamma_0 > -\pi^2/L^2$, for all $x \in [0,L]$ and $q(x) \geq q_0$ such that $q_0 + f'(0) \geq C(L,\gamma_0)L/\pi - (\pi/L)^2$, for all $x \in [0,L]$. Then, there is a unique solution $z \in C([0,T];H^{-1}(0,L)) \cap L^2(0,T;L^2(0,L))$ to (3.7). Furthermore, there exists $C > 0$ such that*

$$\|z\|_{L^\infty(0,T;H^{-1}(0,L))} \leq C\left(\|z_0\|_{H^{-1}(0,L)} + \|g\|_{L^1(0,T;H^{-1}(0,L))} + \|u\|_{L^2(0,T)}\right). \tag{3.37}$$

Proof. From Proposition 3.2.1, the right-hand side of (3.36) defines, for each $z_0 \in H^{-1}(0,L)$, $u \in L^2(0,T)$ and $g \in L^1(0,T;H^{-1}(0,L))$ a linear bounded functional,

$$\Lambda_{z_0,g,u} : h \in L^2(0,T;L^2(0,L)) \longmapsto \Lambda_{z_0,g,u}(h) \in \mathbb{R}. \tag{3.38}$$

We recall here, that w in right-hand side of (3.36), is the unique solution to (3.34), for every $h \in L^2(0,T;L^2(0,L))$. Thus, by the Riesz representation theorem, we get the existence and uniqueness of solution $z \in L^2(0,T;L^2(0,L))$.

Note that due the Proposition 3.2.1, the solution w to (3.34) belongs to the space $C([0,T];H_0^1(0,L)) \cap L^2(0,T;H^2(0,L)\cap H_0^1(0,L))$ even if we choose $h \in L^1(0,T;H_0^1(0,L))$, then $\Lambda_{z_0,g,u}$ also defines a linear bounded functional on $L^1(0,T;H_0^1(0,L))$. Thus by the Riesz Theorem, see [7, Theorem 4.14], we have a unique solution y belonging to $L^\infty(0,T;H^{-1}(0,L))$.

Now we use the Cauchy-Schwartz inequality on (3.36), it holds that

$$\left|\langle z, h\rangle_{L^\infty(0,T;H^{-1}(0,L)),L^1(0,T;H_0^1(0,L))}\right| \leq \|z_0\|_{H^{-1}(0,L)}\|w(0,x)\|_{H_0^1(0,L)} \\ + \|g\|_{L^1(0,T;H^{-1}(0,L))}\|w\|_{L^\infty(0,T;H_0^1(0,L))} + \|u\|_{L^2(0,T)}\|w_x(t,0)\|_{L^2(0,T)}. \quad (3.39)$$

From Proposition 3.2.1 and Corollary 3.2.2, we conclude there exists a constant $C > 0$ such that

$$\left|\langle z, h\rangle_{L^\infty(0,T;H^{-1}(0,L)),L^1(0,T;H_0^1(0,L))}\right| \leq \\ C\left(\|z_0\|_{H^{-1}(0,L)} + \|g\|_{L^1(0,T;H^{-1}(0,L))} + \|u\|_{L^2(0,T)}\right)\|h\|_{L^1(0,T;H_0^1(0,L))}, \quad (3.40)$$

consequently

$$\|z\|_{L^\infty(0,T;H^{-1}(0,L))} \leq C\left(\|z_0\|_{H^{-1}(0,L)} + \|g\|_{L^1(0,T;H^{-1}(0,L))} + \|u\|_{L^2(0,T)}\right). \quad (3.41)$$

Using the above inequality and a density argument we can conclude that z belongs to $C([0,T];H^{-1}(0,L))$. The proof of Proposition 3.2.4 is complete. □

Nonlinear control system (3.6)

Proposition 3.2.5. *Let $f \in W^{2,\infty}(\mathbb{R})$ such that for a positive constant C*

$$f(0) = 0, \quad |f(r_1) - f(r_2)| \leq C\left|r_1^2 - r_2^2\right| \ \forall r_1, r_2 \in \mathbb{R}. \quad (3.42)$$

Then, there exists a positive number $r > 0$ such that for any $z_0 \in H^{-1}(0,L)$, $u \in L^2(0,T)$ and $g \in L^1(0,T;H^{-1}(0,L))$ satisfying

$$\|z_0\|_{H^{-1}(0,L)} + \|u\|_{L^2(0,T)} + \|g\|_{L^1(0,T;H^{-1}(0,L))} \leq r \quad (3.43)$$

the system (3.6) *has a unique solution $z \in C([0,T];H^{-1}(0,L)) \cap L^2(0,T;L^2(0,L))$.*

Proof. Let $z_0 \in H^{-1}(0,L)$, $u \in L^2(0,T)$ and $g \in L^1(0,T;H^{-1}(0,L))$ satisfying (3.43) and let define the following map

$$\mathbf{\Pi} : \ell \in L^2(0,T;L^2(0,L)) \longmapsto \mathbf{\Pi}(\ell) = y \in L^2(0,T;L^2(0,L)) \quad (3.44)$$

where z is solution to

$$\begin{cases} z_t - z_{xx} + q(x)z - F_\gamma(z) = f(\ell) + g, & (t,x) \in (0,T)\times(0,L), \\ z(t,0) = u(t), \quad z(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \quad (3.45)$$

Now, z is a fix point of the map $\mathbf{\Pi}$ if and only if z is a solution to (3.6).

By definition

$$\|f(\ell)\|_{L^1(0,T;H^{-1}(0,L))} = \sup_{w\in L^\infty(0,T;H_0^1(0,L))} \frac{\left|\iint f(\ell)w \,\mathrm{d}x\,\mathrm{d}t\right|}{\|w\|_{L^\infty(0,T;H_0^1(0,L))}}. \quad (3.46)$$

Now, by the a priori estimation on f, see (3.42), we get that

$$\left|\iint f(\ell)w \,\mathrm{d}x\,\mathrm{d}t\right| \leq C\iint \ell^2 |w| \,\mathrm{d}x\,\mathrm{d}t \quad (3.47)$$

for some positive constant C and by the Hölder inequality,

$$\left|\iint f(\ell) w \,\mathrm{d}x\,\mathrm{d}t\right| \leq C\|\ell\|^2_{L^2(0,T;L^2(0,L))}\|w\|_{L^2(0,T;L^\infty(0,L))}, \tag{3.48}$$

then, by the continuous injection of $H^1_0(0,L)$ into $L^\infty(0,L)$, we see that there exists a positive constant C_2 such that

$$\|f(\ell)\|_{L^1(0,T;H^{-1}(0,L))} \leq C_2\|\ell\|^2_{L^2(0,T;L^2(0,L))}. \tag{3.49}$$

Then, by Proposition 3.2.4 and (3.49), we see that exists a positive constant C_3, such that

$$\begin{aligned}&\|\mathbf{\Pi}(\ell)\|_{L^2(0,T;L^2(0,L))} \leq \\ &\qquad C_3\left(\|z_0\|_{H^{-1}(0,L)} + \|u\|_{L^2(0,T)} + \|g\|_{L^1(0,T;H^{-1}(0,L))} + \|\ell\|^2_{L^2(0,T;L^2(0,L))}\right).\end{aligned} \tag{3.50}$$

Let R be a positive number, and define the following set

$$B_R = \left\{\ell^2 \in L^2(0,T;L^2(0,L));\ \|\ell\|_{L^2(0,T;L^2(0,L))} \leq R\right\}. \tag{3.51}$$

Now, in virtue of (3.50), we observe that if r and R are chosen such that $C_3(r+R^2) \leq R$, then

$$\mathbf{\Pi}(B_R) \subset B_R.$$

Let $\bar{\ell}, \ell \in B_R$ and let define $\hat{z} = \mathbf{\Pi}(\bar{\ell}) - \mathbf{\Pi}(\ell)$. It is not difficult to check that $\hat{z}$ satisfies

$$\begin{cases} \hat{z}_t - \hat{z}_{xx} + q(x)\hat{z} - F_\gamma(\hat{z}) = f(\bar{\ell}) - f(\ell), & (t,x) \in (0,T)\times(0,L), \\ \hat{z}(t,0) = 0, \quad \hat{z}(t,L) = 0, & t \in (0,T), \\ \hat{z}(0,x) = 0, & x \in (0,L), \end{cases} \tag{3.52}$$

then we have there exists a postive constant C_3 such that

$$\|\mathbf{\Pi}(\bar{\ell}) - \mathbf{\Pi}(\ell)\|_{L^2(0,T,L^2(0,L))} \leq C_3\|f(\bar{\ell}) - f(\ell)\|_{L^1(0,T;H^{-1}(0,L))}. \tag{3.53}$$

Now, by the assumption (3.42) over f, we get that

$$\left|\iint \left(f(\bar{\ell}) - f(\ell)\right) w \,\mathrm{d}x\,\mathrm{d}t\right| \leq C\iint \left|\bar{\ell}^2 - \ell^2\right| |w| \,\mathrm{d}x\,\mathrm{d}t,\ \forall w \in L^\infty(0,T;H^1_0(0,L)) \tag{3.54}$$

and by the Hölder inequality we can conclude that

$$\begin{aligned}&\iint \left(f(\bar{\ell}) - f(\ell)\right) w \,\mathrm{d}x\,\mathrm{d}t \leq \\ &\qquad C\|\bar{\ell} + \ell\|_{L^2(0,T;L^2(0,L))}\|\bar{\ell} - \ell\|_{L^2(0,T;L^2(0,L))}\|w\|_{L^\infty(0,T;L^\infty(0,L))}.\end{aligned} \tag{3.55}$$

Thus, by the continuous injection of $H^1_0(0,L)$ into $L^\infty(0,L)$, there exists a positive constant C_4 such that

$$\|f(\bar{\ell}) - f(\ell)\|_{L^1(0,T;H^{-1}(0,L))} \leq C_4\|\bar{\ell} + \ell\|_{L^2(0,T;L^2(0,L))}\|\bar{\ell} - \ell\|_{L^2(0,T;L^2(0,L))}. \tag{3.56}$$

Then, for all $\bar{\ell}, \ell \in B_R$, it holds that

$$\|\mathbf{\Pi}(\bar{\ell}) - \mathbf{\Pi}(\ell)\|_{L^2(0,T;L^2(0,L))} \leq 2C_4 R \|\bar{\ell} - \ell\|_{L^2(0,T;L^2(0,L))}. \tag{3.57}$$

Finally, if $2C_4R < 1$, the map $\mathbf{\Pi}$ is a contraction. Thus, in virtue of the Banach fix point Theorem, there exists a unique fix point for the map (3.44) and therefore a unique local solution to equation (3.6). The proof is complete. □

3.2.2 Well-posedness of the control system (3.10)

Proposition 3.2.6. *Let $z_0 \in L^2(0,L)$, $u \in L^2(0,T)$, γ_0 and q_0 such that $\gamma_0 > -\pi^2/L^2$ and $q_0 \geq \frac{1}{\frac{\pi^2}{L^2}+\gamma_0} - (\pi/L)^2$. Then, there is a unique solution $z \in C([0,T];L^2(0,T)) \cap L^2(0,T;H_0^1(0,L))$ to (3.10). Furthermore, there exists $C > 0$ such that*

$$\|z\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H_0^1(0,L))} \leq C(\|u\|_{L^2(0,T)} + \|z_0\|_{L^2(0,L)}). \tag{3.58}$$

Proof. Let $t \in [0,T]$, multiply equation (3.10) by $z(t,x)$ and then perform integrations by parts in $[0,L]$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z^2\,\mathrm{d}x + \int_0^L z_x^2\,\mathrm{d}x + q_0\int_0^L z^2\,\mathrm{d}x = \int_0^L F_{\gamma_0}(z)z\,\mathrm{d}x + \int_0^L u\theta z\,\mathrm{d}x, \tag{3.59}$$

then in virtue of the Poincaré inequality and the continuity of the operator F_{γ_0}, see inequality (3.5), it holds

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z^2\,\mathrm{d}x + \left(\frac{\pi^2}{L^2} + q_0 - \frac{L}{\pi}C(\gamma_0, L)\right)\int_0^L z^2\,\mathrm{d}x \leq \int_0^L u\theta z\,\mathrm{d}x \tag{3.60}$$

then, by the assumption over q_0 and the Hölder inequality applied on the righthand side of the above inequality we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L z^2\,\mathrm{d}x \leq \sqrt{L}\|\theta\|_{L^\infty(0,L)}\,|u|\,\|z\|_{L^2(0,L)}. \tag{3.61}$$

Then, by using the Young's inequality and performing an integration in $t \in [0,T]$, it holds

$$\|z\|^2_{L^2(0,L)} \leq L\|\theta\|^2_{L^\infty(0,L)}\|u\|^2_{L^2(0,T)} + \|z_0\|^2_{L^2(0,L)} + \int_0^T \|z\|^2_{L^2(0,L)}\,\mathrm{d}t,\ \forall t \in [0,T]. \tag{3.62}$$

The Gronwall's lemma allows to us to conclude that there exists a positive constant C_1 such that

$$\|z\|_{L^\infty(0,T;L^2(0,L))} \leq C_1\left(\|u\|_{L^2(0,T)} + \|z_0\|_{L^2(0,L)}\right). \tag{3.63}$$

Now, by an integration in $t \in [0,T]$ of (3.59) and combining with inequality (3.63), it can be conclude that there exists a positive constant C_2 such that

$$\|z_x\|^2_{L^2(0,T;L^2(0,L))} \leq C_2\left(\|u\|_{L^2(0,T)} + \|z_0\|_{L^2(0,L)}\right). \tag{3.64}$$

Finally, (3.58) follows from (3.63) and (3.64). The proof is complete. □

3.3 Controllability

3.3.1 Boundary null controllability of the linear control system (3.7)

In this section we study the boundary null controllability property of the linear control system given by (3.7). To do that, we consider the controllability-observability duality principle.

Let us take a well-posedness framework space (U, X, Y, Z) for the system (3.7), this means that for $u \in U$, $g \in Y$, $z_0 \in X$ there exists a unique $z \in Z$ solution to (3.7). The controllability-observability duality principle says that null controllability of this system it is equivalent to the existence of a positive constant C such that

$$\|w\|_{Y'} + \|w(0,x)\|_{X'} \leq C\left(\|h\|_{Z'} + \|w_x(t,0)\|_{U'}\right) \tag{3.65}$$

For every w solution to

$$\begin{cases} -w_t - w_{xx} + (q - f'(0))w - F_\gamma(w) = h, & (t,x) \in (0,T) \times (0,L), \\ w(t,0) = 0, \quad w(t,L) = 0, & t \in (0,T), \\ w(T,x) = w_T(x), & x \in (0,L), \end{cases} \tag{3.66}$$

with final data $w_T \in X'$ and $h \in Y'$, here the dual spaces are denoted with $'$. The system (3.66) it is called the adjoint system to (3.7). Inequality (3.65) it is called observability inequality for equation (3.66). Let us introduce the followings weight functions

$$\varphi(t,x) := \frac{\beta(x)}{t(T-t)}, \quad \beta(x) = -\left(\frac{x}{L} - 2\right)^2 + 8, \quad (t,x) \in [0,T] \times [0,L]. \tag{3.67}$$

These functions have been previously used in [11].

In this part of the paper we shall use an abbreviated notation for the integrals. We write $\iint$ instead of $\int_0^T \int_0^L$, avoiding the symbols $dx\,dt$ in that case.

Following the procedure described in [32] or [28], we are able to obtain the following Carleman estimate with boundary observation for the adjoint equation (3.66).

Proposition 3.3.1. *There exist constants $C > 0$ and $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ the unique solution $w = w(t,x)$ to the adjoint equation (3.66) with final data $w_T \in H_0^1(0,L)$ and $h \in L^1(0,T; H_0^1(0,L))$ satisfies*

$$\lambda^3 \iint e^{-2\lambda\varphi}\varphi^3 w^2 + \lambda \iint e^{-2\lambda\varphi}\varphi w_x^2 \leq$$
$$C\left(\iint e^{-2\lambda\varphi}(F_\gamma(w) - (q - f'(0))w + h)^2 + \lambda \int_0^T e^{-2\lambda\varphi(t,0)}\varphi_x(t,0)w_x^2(t,0)\,dt\right). \tag{3.68}$$

Proof. See Appendix Section A.6. □

Now in order to obtain a Carleman estimate with the norm of $w(0,x)$ in $H_0^1(0,L)$ in the lefthand side, we introduce the following modified weight function.

Let $0 < T_1 < T$, then we define a new weight function, given by

$$\psi(t,x) = \left\{ \begin{array}{cl} \frac{4}{T^2}\beta(x) & \text{If } (t,x) \in [0,T_1) \times [0,L], \\ \varphi & \text{If } (t,x) \in [T_1,T) \times [0,L]. \end{array} \right. \tag{3.69}$$

With this weight function it holds the following.

Proposition 3.3.2. *Let $0 < T_1 < T_2 < T$, such that there exist positive constants M_1, M_2 such that the solution w to the adjoint equation (3.66) satisfies that*

$$\int_0^{T_1} \|w\|^2_{L^2(0,L)}\, dt \leq M_1 \int_{T_1}^{T_2} \|w\|^2_{L^2(0,L)}\, dt, \quad \int_{T_2}^{T} \|w\|^2_{L^2(0,L)}\, dt \leq M_2 \int_{T_1}^{T_2} \|w\|^2_{L^2(0,L)}\, dt. \tag{3.70}$$

Then there exists a constant $C > 0$ such that for λ large enough it holds

$$\lambda \iint e^{-2\lambda\psi}\psi w_x^2 + \int_0^L w_x^2(0,x)\, dx \leq C\left(\iint h^2 e^{-2\lambda\psi} + \lambda \int_0^T e^{-2\lambda\psi(t,0)}\psi_x(t,0) w_x^2(t,0)\, dt\right) \tag{3.71}$$

for every final data $w_T \in H_0^1(0,L)$ and every h such that $\iint h^2 e^{-2\lambda\psi} < \infty$.

Proof. Let $\eta \in C^\infty(0,T)$ be such that $\eta(t) = 1$ for all $t \in [0,T_1]$ and $\eta(t) = 0$, for all $t \in [T_2,T]$. From energy estimations, and the continuity of operator F_γ it can be checked that the solution w to adjoint system (3.66) satisfies that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L w_x^2\, \mathrm{d}x \leq \frac{3}{2}\|h\|^2_{L^2(0,L)} + \frac{3}{2}\left(\frac{L^2}{\pi^2}C^2(\gamma_0,L) + \|q - f'(0)\|^2_{L^\infty(0,L)}\right)\frac{L^2}{\pi^2}\int_0^L w_x^2\, \mathrm{d}x. \tag{3.72}$$

Multiply (3.72) by η and subtracting $\eta_t \int_0^L w_x^2\, \mathrm{d}x$ from both sides, we obtain that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \eta w_x^2\, \mathrm{d}x \leq \frac{3}{2}\left(\frac{L^2}{\pi^2}C^2(\gamma_0,L) + \|q - f'(0)\|^2_{L^\infty(0,L)}\right)\frac{L^2}{\pi^2}\int_0^L \eta w_x^2\, \mathrm{d}x + \beta(t) \tag{3.73}$$

where $\beta(t) = \frac{3}{2}\int_0^L \eta(t)h^2\, \mathrm{d}x - \eta_t \int_0^L w_x^2\, \mathrm{d}x$.

Performing an integration on $[t,T]$ and recalling that $\eta(T) = 0$, it holds

$$\int_0^L \eta w_x^2\, \mathrm{d}x \leq \frac{3}{2}\left(\frac{L^2}{\pi^2}C^2(\gamma_0,L) + \|q - f'(0)\|^2_{L^\infty(0,L)}\right)\frac{L^2}{\pi^2}\int_0^L \eta(t) w_x(t,x)\, \mathrm{d}x$$

$$+\int_t^T \beta(s)\,\mathrm{d}s \qquad (3.74)$$

and by the Gronwall lemma we obtain that

$$\int_0^L \eta(t) w_x^2(t,x)\,\mathrm{d}x \leq e^{\frac{3}{2}\left(\frac{L^2}{\pi^2}C^2(\gamma_0,L)+\|q-f'(0)\|^2_{L^\infty(0,L)}\right)\frac{TL^2}{\pi^2}} \int_t^T \beta(s)\,\mathrm{d}s, \qquad (3.75)$$

which implies, by definition of η, that there exists a positive constant C_1 such that

$$\|w_x\|^2_{L^\infty(0,T_1;L^2(0,L))} \leq C_1 \left(\|h\|^2_{L^2(0,T_2;L^2(0,L))} + \|w_x\|^2_{L^2(T_1,T_2;L^2(0,L))} \right). \qquad (3.76)$$

Now, from the Carleman inequality (3.68), for λ large enough it holds

$$\left(\lambda^3\left(\frac{16}{T^2}\right)^3 - 2C\|q-f'(0)\|^2_{L^\infty(0,L)}\right) \int_{T_1}^{T_2}\int_0^L e^{-2\lambda\varphi} w^2\,\mathrm{d}x\,\mathrm{d}t + \lambda \iint e^{-2\lambda\varphi}\varphi w_x^2 \leq$$
$$2C\iint e^{-2\lambda\varphi}F_\gamma^2(w) + 2C\iint e^{-2\lambda\varphi}h^2 + C\lambda\int_0^T e^{-2\lambda\varphi(t,0)}\varphi_x(t,0)w_x^2(t,0). \qquad (3.77)$$

Here, we have used that $16/T^2 \leq \varphi(t,x)$ for all $(t,x)\in[0,T]\times[0,L]$.

Thanks to the assumption over T_1 and T_2 in addition to the continuity of the operator F_γ, we obtain that

$$2C\int_0^T\int_0^L e^{-2\lambda\varphi}F_\gamma^2(w)\,\mathrm{d}x\,\mathrm{d}t \leq 6CC^2(\gamma_0,L)\frac{L^2}{\pi^2}\max\{M_1,M_2\}\int_{T_1}^{T_2}\int_0^L w^2\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.78)$$

Let us note that $(t,x)\mapsto e^{-\lambda\varphi(t,x)}$ is a continuous function over $[T_1,T_2]\times[0,L]$, then there exists a positive constant δ, such that $\delta \leq e^{-\lambda\varphi}$. Collecting (3.77) and (3.78) we get

$$\left(\lambda^3\left(\frac{16}{T^2}\right)^3\delta - 2C\|q+f'(0)\|^2_{L^\infty(0,L)}\delta - 6CC^2(\gamma_0,L)\frac{L^2}{\pi^2}\max\{M_1,M_2\}\right)\int_{T_1}^{T_2}\int_0^L e^{-2\lambda\varphi}w^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\lambda\iint e^{-2\lambda\varphi}\varphi w_x^2 \leq 2C\iint e^{-2\lambda\varphi}h^2 + C\lambda\int_0^T e^{-2\lambda\varphi(t,0)}\varphi_x(t,0)w_x^2(t,0). \qquad (3.79)$$

Recalling that $\psi\leq\varphi$ if $t\in[0,T_1]$ and $\psi=\varphi$ for $t\in[T_1,T]$, then for a λ large enough it holds

$$\lambda\int_{T_1}^T\int_0^L e^{-2\lambda\psi}\psi w_x^2\,\mathrm{d}x\,\mathrm{d}t \leq 2C\iint e^{-2\lambda\psi}h^2 + C\lambda\int_0^T e^{-2\psi(t,0)}\psi_x(t,0)w_x^2(t,0)\,\mathrm{d}t. \qquad (3.80)$$

On the one hand, note that $(t,x) \mapsto \lambda e^{-2\lambda\psi}\psi$ is a continuous for $t \in [0,T_1]$ and all $x \in [0,L]$, this implies that there exists a constant $C_2 > 0$ such that

$$\lambda \int_0^{T_1}\int_0^L e^{-2\lambda\psi}\psi w_x^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^L w_x^2(0,x)\,\mathrm{d}x \le C_2 \|w_x\|^2_{L^\infty(0,T_1;L^2(0,L))} \tag{3.81}$$

and by the inequality (3.76), there exists a positive constant C_3 such that

$$\begin{aligned}\lambda \int_0^{T_1}\int_0^L e^{-2\lambda\psi}\psi w_x^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^L w_x^2(0,x)\,\mathrm{d}x \le \\ C_3\left(\|h\|^2_{L^2(0,T_2;L^2(0,L))} + \|w_x\|^2_{L^2(T_1,T_2;L^2(0,L))}\right).\end{aligned} \tag{3.82}$$

Since that $(t,x) \mapsto e^{-2\lambda\psi(t,x)}$ is a strictly positive continuous function on $[0,T_2]\times[0,L]$ and recalling the fact that $16/T^2 \le \psi(t,x)$ for all $(t,x) \in [0,T]\times[0,L]$, it is not difficult to deduce from above inequality that there exists a positive constant C_4 such that

$$\begin{aligned}\lambda \int_0^{T_1}\int_0^L e^{-2\lambda\psi}\psi w_x^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^L w_x^2(0,x)\,\mathrm{d}x \le \\ C_4\left(\int_0^{T_2}\int_0^L e^{-2\lambda\psi}h^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{T_1}^{T_2}\int_0^L e^{-2\lambda\psi}\psi w_x^2\,\mathrm{d}x\,\mathrm{d}t\right).\end{aligned} \tag{3.83}$$

Finally, combining (3.80) and (3.83) we obtain (3.71). The proof is complete. □

Proposition 3.3.3. *Let*

$$\alpha_1 = \frac{\lambda}{T}\min_{x\in[0,L]}\beta(x) = \frac{4\lambda}{T}, \qquad \alpha_2 = \frac{\lambda}{T}\max_{x\in[0,L]}\beta(x) = \frac{7\lambda}{T}. \tag{3.84}$$

There exists $C > 0$ such that the solution w to the adjoint equation (3.66) satisfies

$$\begin{aligned}\max_{t\in[0,T]} \|we^{\frac{-\alpha_2}{T-t}}(T-t)^{3/2}\|^2_{L^\infty(0,L)} + \int_0^L w_x^2(0,x)\,dx \le \\ C\left(\iint h^2 e^{\frac{-2\alpha_1}{T-t}} + \int_0^T w_x^2(t,0)\frac{e^{\frac{-2\alpha_1}{T-t}}}{T-t}\,dt\right)\end{aligned} \tag{3.85}$$

for every $w_T \in H_0^1(0,L)$ and h such that $\displaystyle\iint h^2 e^{\frac{-2\alpha_1}{T-t}} < \infty$.

Proof. Let define $\delta(t) = e^{\frac{-\alpha_2}{T-t}}(T-t)^{3/2}$ and the following change of variable $\tilde{w} = \delta(t)w$, then $\tilde{w}$ satisfies the following equation

$$\begin{cases} -\tilde{w}_t - \tilde{w}_{xx} + (q - f'(0))\tilde{w} - F_\gamma(\tilde{w}) = -\delta_t w + \delta h & (t,x) \in (0,T)\times(0,L), \\ \tilde{w}(t,0) = 0, \quad \tilde{w}(t,L) = 0, & t \in (0,T), \\ \tilde{w}(T,x) = 0, & x \in (0,L). \end{cases} \tag{3.86}$$

By the well-posedness of (3.86), see Proposition 3.2.1, there exists a constant C_1 such that

$$\|\tilde{w}\|^2_{L^\infty(0,T;H_0^1(0,L))} \le C_1\left(\|\delta h\|^2_{L^2(0,T;L^2(0,L))} + \|\delta_t w\|^2_{L^2(0,T;L^2(0,L))}\right). \tag{3.87}$$

We can bound from above the right hand side of (3.87) as follows.

The first term, by the fact that $\alpha_1 < \alpha_2$, it holds

$$\iint h^2 e^{\frac{-2\alpha_2}{T-t}} (T-t)^3 \,\mathrm{d}x\,\mathrm{d}t \le T^3 \iint h^2 e^{\frac{-2\alpha_1}{T-t}} \,\mathrm{d}x\,\mathrm{d}t. \tag{3.88}$$

Now, in order to find an upper bound for the second term in (3.87), notice that there exists a positive constant C_2 such that $\delta_t^2 \le C_2 e^{\frac{-2\alpha_2}{T-t}}(T-t)^{-1}$, for all $t \in [0,T]$. Besides, by the Poincaré inequality, we obtain there exists a positive constant C_3 such that

$$\iint \delta_t^2 w^2 \,\mathrm{d}x\,\mathrm{d}t \le C_3 \iint w_x^2 e^{\frac{-2\alpha_2}{T-t}} (T-t)^{-1} \,\mathrm{d}x\,\mathrm{d}t, \tag{3.89}$$

then, by the continuous injection from $H_0^1(0,L)$ into $L^\infty(0,L)$, there exists a constant C_4, such that

$$\max_{t\in[0,T]} \|we^{-\frac{\alpha_2}{T-t}}(T-t)^{3/2}\|_{L^\infty(0,L)} \le \\ C_4 \left(\iint h^2 e^{\frac{-2\alpha_1}{T-t}} \,\mathrm{d}x\,\mathrm{d}t + \iint w_x^2 e^{\frac{-2\alpha_2}{T-t}} (T-t)^{-1} \,\mathrm{d}x\,\mathrm{d}t \right). \tag{3.90}$$

Finally, combining (3.90) and (3.71) we obtain (3.85). □

Notice that inequality (3.85) can be seen as an observability inequality, in some weighted spaces, as we anticipated in (3.65). Let us precise these spaces, but first we introduce some notation. Let

$$L_t^2(\rho) = \left\{ f : \int_0^T f^2(t)\rho(t)\,\mathrm{d}t < \infty \right\},\ L_{tx}^2(\rho) = \left\{ f : \int_0^T \int_0^L f^2(t,x)\rho(t)\,\mathrm{d}x\,\mathrm{d}t < \infty \right\}. \tag{3.91}$$

In virtue of the previous notation, let call

$$U = L_t^2\left(e^{\frac{2\alpha_2}{T-t}}(T-t)\right),\ X = H^{-1}(0,L),\ Z = L_{tx}^2\left(e^{\frac{2\alpha_1}{T-t}}\right)$$

and

$$Y = \left\{ y :\ (T-t)^{-3/2} e^{\frac{\alpha_2}{T-t}} y \in L^1(0,T;H^{-1}(0,L)) \right\}, \tag{3.92}$$

which define the functional space framework (U,X,Y,Z). The following proposition states the boundary null controllability of the linear system (3.7) in the functional framework given by the spaces (U,X,Y,Z).

Proposition 3.3.4. *For each $g \in Y$ and $z_0 \in H^{-1}$ there exist a control $u \in U$ such that the solution of (3.7) satisfies $z \in L_{tx}^2\left(e^{\frac{2\alpha_1}{T-t}}\right)$. Moreover, the solution belongs to*

$$z \in \mathbf{B} = \left\{ z \in L_{tx}^2\left(e^{\frac{2\alpha_1}{T-t}}\right); (T-t)^2 e^{\frac{\alpha_1}{T-t}} z \in \mathbf{A} \right\} \tag{3.93}$$

and $z(T) = 0$, where $\mathbf{A} = C([0,T];H^{-1}(0,L)) \cap L^2(0,T;L^2(0,L))$.

Proof. In virtue of the controllability-observability duality property and Proposition 3.3.3, we get the existence of the control $u \in U$ and $z \in L_{tx}^2\left(e^{\frac{2\alpha_1}{T-t}}\right)$. Thus, we focus to prove the fact that $(T-t)^2 e^{\frac{\alpha_1}{T-t}} z$ belongs to $C([0,T];H^{-1}(0,L)) \cap L^2(0,T;L^2(0,L))$.

To do that, let define $\hat{z} = (T-t)^2 e^{\frac{\alpha_1}{T-t}} z$, for z solution to (3.7), $\hat{g} = (T-t)^2 e^{\frac{\alpha_1}{T-t}} g$ and $\hat{u} = (T-t)^2 e^{\frac{\alpha_1}{T-t}} u$. It is not difficult to check that $\hat{z}$ satisfies

$$\begin{cases} \hat{z}_t - \hat{z}_{xx} - (q - f'(0))\hat{y} - F_\gamma(\hat{z}) = \hat{g} - 2(T-t)e^{\frac{\alpha_1}{T-t}} z + \alpha_1 e^{\frac{\alpha_1}{T-t}} z & (t,x) \in (0,T) \times (0,L), \\ \hat{z}(t,0) = \hat{u}, \quad \hat{z}(t,L) = 0, & t \in (0,T), \\ \hat{z}(0,x) = T e^{\frac{\alpha_1}{T}} z_0, & x \in (0,L). \end{cases} \tag{3.94}$$

Now, by the regularity of the control, we get that $\hat{u} \in L^2(0,T)$ and using that $z \in L^2_{tx}\left(e^{\frac{2\alpha_1}{T-t}}\right)$ we get that

$$(T-t)e^{\frac{\alpha_1}{T-t}} z \in L^2(0,T; L^2(0,L)) \tag{3.95}$$

which implies that the right-hand side of (3.94) belongs to $L^1(0,T; H^{-1}(0,L))$ and by Proposition 3.2.4 we conclude the $\hat{z} \in \mathbf{A}$. The proof is complete. $\square$

3.3.2 Boundary null controllability of the nonlinear system (3.6)

This section is devoted to the proof the Theorem 3.1.2. To do that, we use a local inversion argument. The core of this kind of argument it is to set a map and a suitable functional framework such that the null controllability property of the linearized system is equivalent to the surjectivity of the linearized map. Thus, Theorem 3.1.2 can be deduced from the following theorem.

Theorem 3.3.5. *Let $\mathbf{E}$ and $\mathbf{G}$ be two Banach spaces and let $\Lambda : \mathbf{E} \to \mathbf{G}$ satisfying $\Lambda \in C^1(\mathbf{E}; \mathbf{G})$. Assume that $\hat{e} \in \mathbf{E}$, $\Lambda(\hat{e}) = \hat{g}$, and $\Lambda'(\hat{e}) : \mathbf{E} \to \mathbf{G}$ is surjective. Then, there exists $r > 0$ such that, for every $g \in \mathbf{G}$ satisfying $\|g - \hat{g}\|_G < r$, there exists some $e \in \mathbf{E}$ solution of the equation $\Lambda(e) = g$.*

The proof of the above theorem can be found in [81], page 107.

As we mention before, the goal of this section is to state the null controllability of the equation (3.6) via a local inversion argument, see Theorem 3.3.5.

Proof of Theorem 3.1.2

Let us set the following spaces

$$\mathbf{E} = \{z \in \mathbf{B} : \mathcal{L}z \in Y\}, \quad \mathbf{G} = H^{-1}(0,L) \times Y. \tag{3.96}$$

where $\mathcal{L}z = z_t - z_{xx} + (q - f'(0))z - F_\gamma(z)$.

Now, let define the following operator

$$\Lambda : \mathbf{E} \longrightarrow \mathbf{G} \tag{3.97}$$

$$z \longmapsto \big(z(0,\cdot), \mathcal{L}z + f'(0)z - f(z)\big) \tag{3.98}$$

The operator Λ is well define if and only if $f(z) - f'(0)z \in Y$ for each $z \in \mathbf{E}$.

Now, $f(z) - f'(0)z \in Y$ is equivalent to

$$\sup_{g \in L^\infty(0,T;H_0^1(0,L))} \frac{\left| \iint \{f(z) - f'(0)z\} (T-t)^{-3/2} e^{\frac{\alpha_2}{T-t}} g \, \mathrm{d}x \, \mathrm{d}t \right|}{\|g\|_{L^\infty(0,T;H_0^1(0,L))}} < \infty, \tag{3.99}$$

Note that $f(z) - f'(0)z = \int_0^1 zf'(sz) - f'(0)z\,\mathrm{d}s$, then for all $f \in W^{2,\infty}(\mathbb{R})$ we have that there exists $C > 0$ such that

$$\left|f(z) - f'(0)z\right| \leq \sup_{s\in[0,1]} \left(f'(sz) - f'(0)\right)|z| \leq Cz^2. \tag{3.100}$$

Now, thanks to the inequality $x \leq e^x$ for all $x \geq 0$ and the fact that $2\alpha_1 - \alpha_2 > 0$, it is not difficult to check that

$$\frac{\left(\frac{2}{3}(2\alpha_1 - \alpha_2)\right)^{\frac{3}{2}}}{(T-t)^{\frac{3}{2}}} \leq e^{\frac{2\alpha_1 - \alpha_2}{T-t}},\ \forall t \in [0,T]. \tag{3.101}$$

In virtue of inequalities (3.100) and (3.101) we obtain that

$$\left|\iint \left\{f(z) - f'(0)z\right\}(T-t)^{-3/2} e^{\frac{\alpha_2}{T-t}} g\,\mathrm{d}x\,\mathrm{d}t\right| \leq \frac{C}{\left(\frac{2}{3}(2\alpha_1 - \alpha_2)\right)^{\frac{3}{2}}} \iint z^2 e^{\frac{2\alpha_1}{T-t}} |g|\,\mathrm{d}x\,\mathrm{d}t \tag{3.102}$$

and by the Hölder inequality and the continuous injection on $H_0^1(0,L)$ into $L^2(0,L)$, there exists a positive constant C_2 such that

$$\left|\iint \left\{f(z) - f'(0)z\right\}(T-t)^{-3/2} e^{\frac{\alpha_2}{T-t}} g\,\mathrm{d}x\,\mathrm{d}t\right| \leq C_2 \|(ze^{\frac{\alpha_1}{T-t}})\|^2_{L^2(0,T;L^2(0,L))} \|g\|_{L^\infty(0,T;H_0^1(0,L))}. \tag{3.103}$$

From inequality (3.103) we can conclude that for $z \in \mathbf{E}$, $f(z) - f'(0)z$ belongs to Y, and therefore Λ is well defined. The continuous injection from $W^{2,\infty}(\mathbb{R})$ into $C^1(\mathbb{R})$, allows to conclude that $\Lambda \in C^1(\mathbf{E};\mathbf{G})$.

Now, the local surjectivity of the operator Λ around zero is equivalent as the local null controllability of the system (3.7). In fact, notice that the functions $z \in \mathbf{E}$ satisfies that $z(T) = 0$ and $\Lambda'(0)$ is given by

$$\Lambda'(0) : \mathbf{E} \to \mathbf{G} \tag{3.104}$$
$$z \longmapsto (z(0,\cdot), \mathcal{L}z)\,. \tag{3.105}$$

Then, the local surjectivity of $\Lambda'(0)$ is equivalent to the null controllability of the linearized equation (3.7) which was proved in Proposition 3.3.4.

Thus, the local null controllability of the equation (3.6) follows from Theorem 3.3.5. The proof of Theorem 3.1.2 is complete. □

3.3.3 Boundary null controllability of system (3.10)

The goal of this section is to prove the boundary null controllability of the system (3.2), which is equivalent to prove that null controllability of the affine control system (3.10). Now, we can observe that if the coefficient $\theta = \xi - F_{\gamma_0}(-\xi_{xx} + \gamma_0\xi)$ on the right-hand side of equation (3.10) satisfies that $\theta(x) \equiv 0$, for all $x \in [0,L]$, the system is not controllable.

Note that if $\xi - F_{\gamma_0}(-\xi_{xx} + \gamma_0\xi) \equiv 0$ for all $x \in [0,L]$, then $\xi = F_{\gamma_0}(-\xi_{xx} + \gamma_0\xi)$ and therefore, by definition of the operator F_{γ_0}, ξ has to satisfy the following boundary value problem

$$\begin{cases} -\xi_{xx} + \gamma_0 \xi = -\xi_{xx} + \gamma_0 \xi, & x \in (0, L), \\ \xi(0) = 0, \quad \xi(L) = 0, \end{cases} \tag{3.106}$$

which implies that $\xi \equiv 0$ and therefore give a contradiction. So $\xi - F_{\gamma_0}(-\xi_{xx} + \gamma_0 - \xi) \neq 0$ for at least one non empty open subset $\omega \subset [0, L]$. Moreover, fix $\theta = \xi - F_{\gamma_0}(-\xi_{xx} + \gamma_0 \xi)$, then we obtain that $F_\gamma(-\xi_{xx} + \gamma_0 \xi) = \xi - \theta$, which implies that $\xi - \theta$ must satisfy the following boundary value problem

$$\begin{cases} -(\xi - \theta)_{xx} + \gamma_0(\xi - \theta) = -\xi_{xx} + \gamma_0 \xi, & x \in (0, L), \\ (\xi - \theta)(0) = 0, \ (\xi - \theta)(L) = 0. \end{cases} \tag{3.107}$$

Then, θ should be solution to

$$\begin{cases} -\theta_{xx} + \gamma_0 \theta = 0, & x \in (0, L), \\ \theta(0) = 1, \ \theta(L) = 0. \end{cases} \tag{3.108}$$

Moreover, θ is given by

$$\theta(x) = \operatorname{csch}\left(L\sqrt{\gamma_0}\right) \sinh\left((L - x)\sqrt{\gamma_0}\right). \tag{3.109}$$

From above equation we conclude that $\theta \neq 0$, independent from the choice of the lift function ξ.

Let us begin with the following characterization's lemma of null controllability property for equation (3.10).

Lemma 3.3.6. *The control system (3.10) is null controllable in $T > 0$ if and only if for any $z_0 \in L^2(0, L)$ there is a function $u \in L^2(0, T)$ such that for any $w_T \in L^2(0, L)$ it holds*

$$\int_0^L z(0, x) w(0, x)\, dx = -\int_0^T \int_0^L u(t)\theta(x) w(t, x)\, dx\, dt. \tag{3.110}$$

where θ is solution to (3.108) and $w(t, x)$ is solution to

$$\begin{cases} -w_t - w_{xx} + q_0 w - F_{\gamma_0}(w) = 0, & (t, x) \in (0, T) \times (0, L), \\ w(t, 0) = 0, \quad w(t, L) = 0, & t \in (0, T), \\ w(T, x) = w_T(x), & x \in (0, L). \end{cases} \tag{3.111}$$

Proof. Consider equation (3.10) and multiply by w solution to (3.111), then by performing an integration by parts we get the following equality

$$\int_0^L z(T, x) w(T, x)\, \mathrm{d}x - \int_0^L z(0, x) w(0, x)\, \mathrm{d}x = \int_0^T \int_0^L u(t)\theta(x) w(t, x)\, \mathrm{d}x\, \mathrm{d}t. \tag{3.112}$$

If (3.110) holds, then $\int_0^T z(T, x) w_T\, \mathrm{d}x = 0$ for all $w_T \in L^2(0, L)$ and therefore $z(T) = 0$. Now if u drives the solution z to (3.10) from z_0 to $z(T, x) = 0$, then from (3.112) we get (3.110). The proof is complete. $\square$

Now, the underlying spatial operator associated to the equation (3.10) is given by

$$\begin{aligned} A_0 : D(A_0) \subset L^2(0,L) &\longrightarrow L^2(0,L), \\ v &\longmapsto A_0 v = -v_{xx} + q_0 v - F_{\gamma_0}(v). \end{aligned} \tag{3.113}$$

The following lemma give us the spectral analysis of operator A_0

Lemma 3.3.7. *Let $\gamma_0 > -\pi^2/L^2$ and $q_0 > 1/\left(\frac{\pi^2}{L} + \gamma_0\right) - \frac{\pi^2}{L^2}$. The eigenvalues and eigenfunctions of operator A_0 are the following*

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2 + q_0 - \frac{1}{\left(\frac{k\pi}{L}\right)^2 + \gamma_0}, \tag{3.114}$$

$$\phi_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right), \ \forall x \in [0,L]. \tag{3.115}$$

Proof. Consider the operator F_{γ_0}, given by (3.3). Let us consider the eigenvalues and eigenfunctions $(\delta_k, \phi_k)_k$ $k \in \mathbb{N}$, associated to F_{γ_0}, that is $F_{\gamma_0}(\phi_k) = \delta_k \phi_k$, which implies that $\delta_k v_k$, has to satisfies the following boundary problem

$$\begin{cases} -(\delta_k v_k)'' + \gamma_0(\delta_k v_k) = v_k, \ \forall k \in \mathbb{N} \\ \delta_k \phi_k(0) = 0, \ \ \delta_k \phi_k(L) = 0 \end{cases} \tag{3.116}$$

Then, $\phi_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right)$, and $\delta_k = \frac{1}{\left(\frac{k\pi}{L}\right)^2 + \gamma_0}$. Now, Let us define the operator $B_0 = A_0 + F_{\gamma_0}$, then

$$B_0 v = -v'' + q_0 v, \tag{3.117}$$

with boundary conditions $v(0) = v(L) = 0$. Let (μ_k, ψ_k) be the eigenvalues and eigenfunctions of operator B_0, which ones are given by

$$\mu_k = \left(\frac{k\pi}{L}\right)^2 + q_0 \tag{3.118}$$

$$\psi_k = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) \tag{3.119}$$

Since $\phi_k = \psi_k$, for all $k \in \mathbb{N}$. We get that, for all $k \in \mathbb{N}$

$$A_0 \phi_k = (\mu_k - \delta_k)\, \phi_k. \tag{3.120}$$

Then, the eigenvalues and eigenfunctions of operator A_0 are given by equations (3.114) and (3.115). The proof of Lemma 3.3.7 is completed. $\square$

Since that ϕ_k is orthonormal basis of $L^2(0,L)$. Let $w_T \in L^2(0,L)$, then $w_T = \sum_k w_{Tk} \phi_k$ and the solution w to (3.111) is given by

$$w(t,x) = \sum_k w_{Tk} e^{-(T-t)\lambda_k} \phi_k(x). \tag{3.121}$$

Using this fact in (3.112), we get the following lemma

Lemma 3.3.8. *The control system (3.10) is null controllable in time $T > 0$ if and only if for any*

$$z_0 = \sum_k z_{0k}\phi_k \in L^2(0, L) \tag{3.122}$$

there exists a function $f(\cdot) \in L^2(0, T)$ such that

$$\int_0^T f(t)e^{-t\lambda_k}\, dt = -\frac{z_{0k}e^{-T\lambda_k}}{\theta_k} \quad \forall k \in \mathbb{N}, \tag{3.123}$$

where

$$\theta_k = \sqrt{\frac{2}{L}}\,\mathrm{csch}(L\sqrt{\gamma_0})\sinh\left(L\sqrt{\gamma_0}\right)\frac{kL\pi}{k^2\pi^2 + L^2\gamma_0}, \quad k \in \mathbb{N} \tag{3.124}$$

are the Fourier coefficients of the expansion $\theta(x) = \sum_k \theta_k\phi_k(x)$. The control is given by $u(t) = f(T - t)$.

Remark 3.3.9. *From equation (3.124), we can see that $\theta_k \neq 0$ for all $k \in N$.*

Proof. The equation (3.123) follows directly from combining (3.121) and (3.122) with (3.110). Thus,

$$\int_0^L \sum_k z_{0k}\phi_k \sum_j e^{-T\lambda_j}\phi_j\, \mathrm{d}x = -\int_0^T\int_0^L u(t)\sum_k \theta_k\phi_k \sum_j e^{-(T-t)\lambda_j}\phi_j\, \mathrm{d}x\, \mathrm{d}t, \tag{3.125}$$

by using the orthogonality of ϕ_k in $L^2(0, L)$, we get

$$-\int_0^T u(t)e^{-(T-t)\lambda_k} = \frac{z_{0k}e^{-T\lambda_k}}{\theta_k}, \quad \forall k \in \mathbb{N}, \tag{3.126}$$

and by the change of variable $T - t \mapsto t$ we get

$$\int_0^T u(T-t)e^{-t\lambda_k} = -\frac{z_{0k}e^{-T\lambda_k}}{\theta_k}, \quad \forall k \in \mathbb{N}. \tag{3.127}$$

Finally, put $f(t) = u(T - t)$ to obtain (3.3.8). The proof is complete. □

Proof of Theorem 3.1.3

As we stated through Lemma 3.3.8, the boundary null controllability of the system (3.2) is equivalent to prove the solvability of the problem of moments (3.123).

Let λ_k the eigenvalues of A_0 defined by (3.114) and $\Lambda = \left(e^{-\lambda_k t}\right)_{k\geq 1}$, $k \in \mathbb{N}$ be the corresponding exponential real family.

Proposition 3.3.10. *Let $\gamma_0 > -\pi^2/L^2$ and $q_0 > 1/\left(\frac{\pi^2}{L} + \gamma_0\right) - \frac{\pi^2}{L^2}$ and $T > 0$. There exists a biorthogonal sequence $(p_m(\cdot))_{m\geq 1}$, $k \in \mathbb{N}$ to the family Λ in $L^2(0, T)$. That is*

$$\int_0^T e^{-\lambda_i t}p_j(t)\, dt = \delta_{ij}, \ \forall\, (i, n) \in \mathbb{N}. \tag{3.128}$$

Proof. Accordingly to the Münz Theorem, see theorem 2.6.3 in [59], the space generated for the family Λ is complete in $L^2(0,T)$, for any $T>0$, if and only if it holds

$$\sum_k \frac{1}{\lambda_k} = \infty. \tag{3.129}$$

In our case, we have that

$$\begin{aligned}\sum_k \frac{1}{\lambda_k} &= \sum_k \frac{1}{\left(\frac{k\pi}{L}\right)^2 + q_0 - \frac{1}{\left(\frac{k\pi}{L}\right)^2+\gamma_0}},\\ &\leq \sum_k \frac{1}{\left(\frac{k\pi}{L}\right)^2 + q_0 - \frac{1}{\left(\frac{\pi}{L}\right)^2+\gamma_0}}.\end{aligned}$$

By an integral criterion we can conclude that $\sum_k \frac{1}{\lambda_k}$ is finite. Then, the space spanned by Λ is a proper space of $L^2(0,T)$ for any $T>0$.

Then, using Theorem 2.6.4 in [59], we can conclude the existence of a biorthogonal sequence $(p_m(t))_{m\geq 1}$ of minimal norm. □

Proposition 3.3.11. *Let $(p_m(\cdot))_{m\geq 1}$ be the biorthogonal sequence given by Proposition 3.3.10. There exist positive constants M and ω, independent of T such that,*

$$\|p_m(\cdot)\|_{L^2(0,T)} \leq M e^{\omega\sqrt{\lambda_k}}. \tag{3.130}$$

Proof. Now, the goal is to obtain the L^2 estimate (3.130). To do that, we study first the case when $T=\infty$ and then the case $T<\infty$.

Let us begin introducing some useful notation for this part.

For any $T>0$, let $E(\Lambda,T)$ the space generated by Λ in $L^2(0,T)$, and $E(m,\Lambda,T)$ the subspace generated by $\left(e^{-\lambda_k t}\right)_{k\geq 1,\, k\neq m}$.

L^2 estimation with $T=\infty$

Let $E^n(\Lambda,\infty)$ be the finite dimensional space generated by $\left(e^{-\lambda_k t}\right)_{1\leq k\leq n}$ in $L^2(0,\infty)$ and let $E^n(m,\Lambda,\infty)$ be the finite dimensional space generated by $\left(e^{-\lambda_k t}\right)_{1\leq k\leq n,\, k\neq m}$ in $L^2(0,\infty)$.

In one hand, for any $n\geq 1$, there exists a unique biorthogonal family $(p_m^n)_{1\leq m\leq n}\subset E^n(\Lambda,T)$, to the family of exponentials $\left(e^{-\lambda_k t}\right)_{1\leq k\leq n}$. Then for $1\leq m,l\leq n$ it holds

$$\int_0^\infty p_m^n(t) e^{-\lambda_l t}\,\mathrm{d}t = \delta_{m,l}. \tag{3.131}$$

On the other hand, any element of the sequence (p_m^n) belongs to $E^n(\Lambda,T)$. We recall that $E^n(\Lambda,T)$ is a finite dimensional space, then (p_m^n) is a finite linear combination of exponentials , that is

$$p_m^n = \sum_{k=1}^n c_m^k e^{-\lambda_k t} \tag{3.132}$$

Then, combining (3.131) and (3.132), we get that

$$\|p_m^n(\cdot)\|^2_{L^2(0,\infty)} = \sum_{l=1}^{n} c_m^l \int_0^\infty e^{-\lambda_l t} p_m^n(t)\,\mathrm{d}t = \sum_{l=1}^{n} c_m^l \delta_{m,l} = c_m^m. \tag{3.133}$$

Using again the orthogonality of p_m^n respect to $e^{-\lambda_l t}$, it holds that

$$\delta_{m,l} = \sum_{k=1}^{m} c_m^k \int_0^\infty e^{-\lambda_k t} e^{-\lambda_l t}\,\mathrm{d}t,\ 1 \leq m, l \leq n. \tag{3.134}$$

If G denotes the Gram matrix of the family Λ. That is, the matrix of elements

$$g_k^l = \int_0^\infty e^{-\lambda_l t} e^{-\lambda_k t}\,\mathrm{d}t = \frac{1}{\lambda_k + \lambda_l},\ 1 \leq k, l \leq n. \tag{3.135}$$

Then, from (3.134) we observe that c_m^k are the elements of the inverse of G. By the Cramer's rule we get that

$$c_m^m = \frac{|G_m|}{|G|}, \tag{3.136}$$

where $|G|$ is the determinant of the matrix G and $|G_m|$ is the determinant of the matrix G_m obtained by changing the m-th column of G my the m-th vector of the canonical basis. Then, it follows, from (3.133) and (3.136) that

$$\|p_m^n(\cdot)\|^2_{L^2(0,\infty)} = \sqrt{\frac{|G_m|}{|G|}}. \tag{3.137}$$

Using the Lemma 2.6.2 in [59], it holds that

$$\frac{|G_m|}{|G|} = 2\lambda_m \prod_{k=1, k\neq m}^{n} \frac{(\lambda_m + \lambda_k)^2}{(\lambda_m - \lambda_k)^2}. \tag{3.138}$$

Then, it can be deduce that

$$\|p_m^n\|_{L^2(0,\infty)} = \sqrt{2}\sqrt{\lambda_m} \prod_{k=1, k\neq m}^{n} \frac{(\lambda_m + \lambda_k)}{|\lambda_m - \lambda_k|}. \tag{3.139}$$

Lemma 3.3.12. *The norm of the biorthogonal sequence $(p_m(\cdot))_{m\geq 1}$ to the family Λ in $L^2(0,\infty)$ given by Proposition 3.3.10 satisfies that, for any $m \geq 1$*

$$\|p_m(\cdot)\|_{L^2(0,\infty)} = \sqrt{2}\sqrt{\lambda_m} \prod_{k=1, k\neq m}^{\infty} \frac{(\lambda_m + \lambda_k)}{|\lambda_m - \lambda_k|}. \tag{3.140}$$

The proof of Lemma (3.3.12) can be found in [59, p. 146].

Lemma 3.3.13. *Let be $\gamma_0 \geq 0$ and $q_0 \geq \frac{1}{\left(\frac{\pi}{L}\right)^2+\gamma_0}$. There exist constants M and ω such that for any λ_m, $m \geq 1$,*

$$\prod_{k=1,k\neq m}^{\infty} \frac{(\lambda_m+\lambda_k)}{|\lambda_m-\lambda_k|} \leq Me^{\omega\sqrt{\lambda_m}} \tag{3.141}$$

Proof. See Appendix A.7. □

Finally combining Lemma 3.3.12 and Lemma 3.3.13 and taking into account that $\sqrt{\lambda_m} \leq e^{\sqrt{\lambda_m}}$ it can be conclude that

$$\|p_m(\cdot)\|_{L^2(0,\infty)} \leq Me^{(\omega+1)\sqrt{\lambda_m}}. \tag{3.142}$$

L^2 estimate for $T < \infty$

In order to estimate the norm of the biothtogonal sequence, given by Proposition 3.3.10, in $L^2(0,T)$ we use the following result

Proposition 3.3.14. *Let Λ be the family of exponentials functions $\left(e^{-\lambda_k t}\right)_{k\geq 1}$ and let $T > 0$. Then, the restriction operator*

$$R_T : E(\Lambda,\infty) \to E(\Lambda,T) \tag{3.143}$$

$$z \mapsto R_T(z) = z|_{[0,T]} \tag{3.144}$$

is invertible and there exists a constant $C > 0$, which only depends on T, such that

$$\|R_T^{-1}\| \leq C. \tag{3.145}$$

The proof of the previous proposition can be found in [59, Theorem 2.6.7 p. 149].

Let $(p_m(\cdot))_{m\geq 1}$ be the biorthogonal family to Λ in $L^2(0,\infty)$ and let $(\tilde{p}_m(\cdot))_{m\geq 1}$ be the biorthogonal family to Λ in $L^2(0,T)$.

Let $(R_T^{-1})^* : E(\Lambda,\infty) \to E(\Lambda,T)$ be the adjoint of R_T^{-1}, then

$$\delta_{k,m} = \int_0^\infty e^{-\lambda_k t} p_m(t)\,\mathrm{d}t = \int_0^\infty (R_T^{-1}R_T)(e^{-\lambda_k t})p_m(t)\,\mathrm{d}t \tag{3.146}$$

$$= \int_0^T R_T(e^{-\lambda_k t})(R_T^{-1})^*(p_m(t))\,\mathrm{d}t \tag{3.147}$$

Now, $(R_T^{-1})^*(p_m(t)) \in E(\Lambda,T)$, and the uniqueness of the biorthogonal sequence we get that

$$(R_T^{-1}(p_m(t))) = \tilde{p}_m(t),\ \forall m \geq 1 \tag{3.148}$$

Finally, to evaluate the L^2 norm in $(0,T)$ consider

$$\|\tilde{p}_m(\cdot)\|_{L^2(0,T)} = \|(R_T^{-1})^*(p_m(\cdot))\|_{L^2(0,T)} \leq \|R_T^{-1}\|\|p_m(\cdot)\|_{L^2(0,\infty)}. \tag{3.149}$$

The proof of Proposition 3.3.11 is completed. □

Using the biorthogonal sequence $(p_m(t))_{m\geq 1}$, $m \in N$ we can consider the function f defined as follows

$$f(t) = \sum_m -\frac{z_{0m}}{\theta_m} e^{-T\lambda_m} p_m(t) \tag{3.150}$$

Then, replacing f in (3.123), formally we get that

$$\int_0^T f(t)e^{-\lambda_k t}\,\mathrm{d}t = \sum_m -\frac{z_{0m}}{\theta_m} e^{-T\lambda_m} \int_0^T p_m(t)e^{-\lambda_k t}\,\mathrm{d}t. \tag{3.151}$$

thus, by the biorthogonality of $(p_m(t))_{m\geq 1}$, we get that

$$\int_0^T f(t)e^{-\lambda_k t}\,\mathrm{d}t = -\frac{z_{0k}}{\theta_k} e^{-T\lambda_k},\ \forall k \in \mathbb{N}. \tag{3.152}$$

Finally, we estimate the L^2 norm of f, using the estimation of the L^2 norm for $p_m(\cdot)$, see inequality (3.130) given by Proposition 3.3.10. Thus,

$$\|f(\cdot)\|_{L^2(0,T)} \leq \sum_k \left|\frac{z_{0k}}{\theta_k}\right| e^{-T\lambda_k} \|p_k\|_{L^2(0,T)}, \tag{3.153}$$

$$\leq M \sum_k \left|\frac{z_{0k}}{\theta_k}\right| e^{-T\lambda_k + \omega\sqrt{\lambda_k}} < \infty \tag{3.154}$$

The proof of Theorem 3.1.3 is completed. □

Chapter 4

Stabilization of a heat equation under disturbance

This chapter is currently under preparation for a submission. This is a joint work with Patricio Guzmán[1].

Summary

4.1 Introduction

Once a mathematical model is established, one relevant task in control theory, among many others, is to design feedback laws to stabilize the state of the system to their equilibria or to another state of interest. In the vast literature of stabilization we can frequently find that mathematical models are analyzed under ideal assumptions in which disturbances are neglected for the sake of simplicity. However, it is known that disturbances are always present and indeed correspond to an additional source of instability. These two facts constitute the main motivation to include the effects of disturbances into the stabilization analysis.

Let $L \in (0,\infty)$ and $a \in C^1([0,L])$. Let us consider

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty)\times(0,L), \\ z_x(t,0) = 0, & t \in (0,\infty), \\ z_x(t,L) = u(t) + d(t), & t \in (0,\infty), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{4.1}$$

In (4.1) the state of the system is denoted by $z = z(t,x)$, the boundary feedback law by $u(t)$ and the unknown boundary disturbance by $d(t)$.

As far as the undisturbed case is concerned, which is when the disturbance is zero, the sources of instability of (4.1) are its boundary conditions and $a^+(x) = \max\{a(x), 0\}$ (the non-negative part of a). In that case the rapid stabilization problem for (4.1) has been successfully solved in [55] with a boundary feedback law designed by means of the backstepping method and Lyapunov techniques. Such a feedback law

[1]Contact: *patricio.guzmanm@usm.cl*, Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile.

is given by [55, equation (3.3)] and reads as

$$u(t) = -k(L,L)z(t,L) - \int_0^L k_x(L,s)z(t,s)\,\mathrm{d}s, \tag{4.2}$$

where $k = k(x,s)$ is a C^2 function on the triangle $\Omega = \{(x,s) \in \mathbb{R}^2 \ / \ 0 \le s \le x \le L\}$ being the unique solution to

$$\begin{cases} k_{xx}(x,s) - k_{ss}(x,s) = (a(s)+\omega)k(x,s), & (x,s) \in \Omega, \\ k_s(x,0) = 0, & x \in [0,L], \\ k(x,x) = \frac{1}{2}\int_0^x (a(s)+\omega)\,\mathrm{d}s, & x \in [0,L], \end{cases} \tag{4.3}$$

where $\omega > 0$ is a constant which fixes the rate for the exponential decay of the target system

However, in the disturbed case it is uncertain whether we can employ (4.2), since in the construction of the gain kernel, based on the application of the method of successive approximations, see for instance [49, Chapter 4], to solve (4.3), no information of the disturbance is used, and hence, (4.2) might not be able to handle the effects of it. Accordingly, in (4.1) we may regard the disturbance as another source of instability and a new boundary feedback law is required to solve the problem under consideration.

In the recent decades, many control approaches have been developed in order to deal with uncertainties in PDE control systems. For example, in [68], the principle of internal model has been implemented to reject the disturbance generated by an exosystem. In [36], an adaptive control method is used to stabilize a wave equation under harmonic disturbance. In the same direction, in [37] the authors stabilized an Euler-Bernoulli beam equation under harmonic disturbance.

Accordingly [54], the authors pointed that when uncertainties modify the PDE system through the boundaries or in-domain dynamics. That is, boundary or distributed external disturbances or unknown parameters are present in the equation. There are, at least, three types of methods to deal with the stabilization problem in presence of an external disturbance or an unknown parameters,

- *Adaptive control*, see for instance [1],
- *Sliding mode control*, see for instance [54],
- *Active disturbance rejection control (ADRC)*, see [85, 25].

The main idea of the disturbance rejection control, is to propose an extended state observer to estimate both the state and the disturbance and then cancel off the disturbance via a stabilizing feedback control law. That flexibility has allowed to apply this method in different context. For instance, in [86], the authors deal with the stabilization of 1-d unstable wave equation. In [53], for instance, the authors solve an output feedback tracking problem for a stable heat equation under boundary disturbances, based in the ADRC method.

The feedback design that we proposed in this work cancel the effects of the disturbances not by using an estimation of the disturbance but using in a suitable way the multivalued operator sign$(\cdot)$, defined by sign : $\mathbb{R} \to 2^{\mathbb{R}}$

$$\operatorname{sign}(f) = \begin{cases} \frac{f}{|f|} & \text{if} \quad f \neq 0, \\ [-1,1] & \text{if} \quad f = 0. \end{cases} \tag{4.4}$$

where $2^{\mathbb{R}}$, denotes the power set of $\mathbb{R}$.

Even if in our analysis we consider an unknown boundary disturbance, we still need to establish some basic assumptions, which are:

(A1) There exists $D \in (0, \infty)$ such that

$$|d(t)| \leq D, \quad \forall t \in [0, \infty) \tag{4.5}$$

(A2) The disturbance d satisfies the following regularity assumption

$$d \in W^{2,1}(0, \infty) \text{ and } d(0) = 0. \tag{4.6}$$

The Assumption **(A1)** is required for the design of a boundary feedback law able to handle the effects of an unknown boundary disturbance while **(A2)** is required for the proof of the well-posedness of the corresponding closed-loop system.

Under similar assumptions, as (4.5) and (4.6), in Table 4.1, we collect some works where the authors solved the problem of stabilization for some PDEs under the influence of unknown disturbances. The disturbance may act either in the domain or at the boundary.

Equation	Distributed disturbance	Boundary disturbance
Heat	[38]	[46]
Wave	[31]	[33, 60]
Beam	[2]	[34, 44, 50]
Schrödinger		[47]

TABLE 4.1: Stabilization of PDEs with unknown disturbances meeting similar assumptions to **(A1)** and **(A2)**.

All these works are of one-dimensional nature, except [38]. In all these works the effects of unknown disturbances were handled with the aid of the sign$(\cdot)$ multivalued operator defined in (4.22), by properly including it in the design of the feedback laws.

4.1.1 Problem statement and main results

In this work we address the rapid stabilization problem for an unstable heat equation with an unknown boundary disturbance. In other words, is to design a boundary feedback law so that the corresponding closed-loop system is exponentially stable in $L^2(0, L)$, with decay rate as large as desired. The main result of this work, is summarized as follows.

Theorem 4.1.1. *Let $a \in C^1([0, L])$, $\omega > 0$. Let us assume **(A1)** and **(A2)**. Let $k = k(x, s)$ be the gain kernel obtained from (4.3). For a regular enough function $f = f(t, x)$ let us introduce the boundary feedback law*

$$u(t,f) = -k(L,L)f(t,L) - \int_0^L k_x(L,s)f(t,s)\,ds \\ - D sign\left(f(t,L) + \int_0^L k(L,s)f(t,s)\,ds\right). \quad (4.7)$$

Let us take an initial condition z_0 in the following set

$$\left\{z_0 \in H^2(0,L),\ \text{such that}\ \ y_0'(0) = 0 \text{ and } y_0'(L) + D sign(y_0(L)) \ni 0\right\}, \quad (4.8)$$

where we have introduced

$$y_0(x) = z_0(x) + \int_0^x k(x,s)z_0(s)\,ds. \quad (4.9)$$

Then, there exists a unique $z = z(t,x)$ in $W^{1,1}(0,\infty; L^2(0,L)) \cap L^1(0,\infty; H^2(0,L))$ such that

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty)\times(0,L), \\ z_x(t,0) = 0 & t \in [0,\infty), \\ z_x(t,L) \ni u(t,z) + d(t), & t \in [0,\infty), \\ z(0,x) = z_0(x), & x \in [0,L]. \end{cases} \quad (4.10)$$

Moreover, (4.10) is exponentially stable in $L^2(0,L)$, with decay rate given by ω. In other words, given $\omega > 0$ the solution z to the closed-loop system (4.10) satisfies

$$\|z(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|z_0\|_{L^2(0,L)}, \quad \forall t \in [0,\infty). \quad (4.11)$$

The boundary feedback law (4.7) is composed by two parts: the role of the first part, which actually is (4.2), is to achieve the desired decay rate while the role of the second part, in which the sign multivalued operator is involved, is to handle the effects of the boundary unknown disturbance. Overall, (4.7) is designed by means of the backstepping method and Lyapunov techniques. The backstepping method, see for instance [49], has shown to be useful for solving the rapid stabilization problem, as can be consulted in [55, 77, 76, 15, 16, 47] for instance.

Let us also note that (4.10), the closed-loop system is no longer a PDE but a differential inclusion due to the presence of the sign$(\cdot)$ multivalued operator. Thus, the maximal monotone operator theory, see [6, 75, 5] for instance, is adequate for studying its well-posedness.

4.1.2 Organization

This chapter is organized as follows. Section 4.2 is dedicated to the feedback design. Here we explain how to combine the backstepping method, the multivalued operator sign$(\cdot)$ and Assumption **(A1)** in order to achieve a feedback law that stabilizes the system and meanwhile rejects the effects of the disturbance. Section 4.3 is devoted to the proof of the main result of this chapter, Theorem 4.1.1, that is, the well-posedness of the resulting closed-loop system and the exponential decay of the its solution. Finally, in Section 4.4, numerical simulations are presented in order to illustrate our theoretical result.

4.2 Feedback design

The main idea behind the feedback design proposed here is to split the control as follows

$$u(t) = u_1(t) + u_2(t), \tag{4.12}$$

where u_1 is the part of the control which deals with the instability caused by the positive part of the coefficient a in equation (4.1) and u_2 part is designed to reject the effects of the unknown disturbance d.

Let us begin with the design of u_1. Consider the backstepping transformation given by

$$v(x) = z(x) + \int_0^x k(x,s)z(s)\,\mathrm{d}s, \tag{4.13}$$

where gain kernel $k = k(x,s)$ is a C^2 function solution to

$$\begin{cases} k_{xx}(x,s) - k_{ss}(x,s) = (a(s)+\omega)k(x,s), & (x,s) \in \Omega, \\ k_s(x,0) = 0, & x \in [0,L], \\ k(x,x) = \frac{1}{2}\int_0^x (a(s)+\omega)\,\mathrm{d}s, & x \in [0,L]. \end{cases} \tag{4.14}$$

Here, $\Omega = \left\{(x,s) \in \mathbb{R}^2 \;/\; 0 \le s \le x \le L\right\}$ and $\omega > 0$ is a constant, which can be fixed as large as desired. More details can be found in [55].

Let us note that the transformation (4.13) maps the system

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty) \times (0,L), \\ z_x(t,0) = 0, & t \in (0,\infty), \\ z_x(t,L) + k(L,L)z(t,L) + \int_0^L k_x(L,s)z(t,s)\,\mathrm{d}s = 0, & t \in (0,\infty), \\ z(0,x) = z_0, & x \in (0,L), \end{cases} \tag{4.15}$$

into the exponentially stable target system

$$\begin{cases} v_t - v_{xx} = -\omega v, & (t,x) \in (0,\infty) \times (0,L), \\ v_x(t,0) = 0, & t \in (0,\infty), \\ v_x(t,L) = 0, & t \in (0,\infty), \\ v(0,x) = v_0, & x \in (0,L). \end{cases} \tag{4.16}$$

We recall that k is the solution to kernel equation (4.14). Then, we choose u_1 as follows

$$u_1(t) = k(L,L)z(t,L) + \int_0^L k_x(L,s)z(t,s)\,\mathrm{d}s. \tag{4.17}$$

Now, we focus on the design of u_2. To do that, we plug-in u_1 into the control system (4.1) and using the transformation (4.13), we obtain the following system

$$\begin{cases} v_t - v_{xx} = -\omega v, & (t,x) \in (0,\infty) \times (0,L), \\ v_x(t,0) = 0, & t \in (0,\infty), \\ v_x(t,L) = u_2(t) + d(t), & t \in (0,\infty), \\ v(0,x) = v_0, & x \in (0,L). \end{cases} \tag{4.18}$$

We multiply (4.18) by y and then we perform an integration by parts, to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2(0,L)} + \omega\|v\|^2_{L^2(0,L)} = -\|v_x\|^2_{L^2(0,L)} + v_x(t,L)v(t,L),\ \forall t \geq 0. \tag{4.19}$$

By the boundary condition at $x = L$ in (4.18), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2(0,L)} + \omega\|v\|^2_{L^2(0,L)} = -\|v_x\|^2_{L^2(0,L)} + (u_2(t) + d(t))v(t,L), \tag{4.20}$$

Here, the idea is to choose u_2 in a suitable way, in order to obtain that the right-hand side of (4.20) be negative. This leads to choose u_2 as follows

$$u_2(t) = -D\mathrm{sign}(v(t,L)), \tag{4.21}$$

where the operator $\mathrm{sign}(\cdot)$ is given by $\mathrm{sign} : \mathbb{R} \to 2^{\mathbb{R}}$, and $2^{\mathbb{R}}$ denotes the power set of $\mathbb{R}$

$$\mathrm{sign}(f) = \begin{cases} \frac{f}{|f|} & \text{if} \quad f \neq 0, \\ \\ [-1,1] & \text{if} \quad f = 0. \end{cases} \tag{4.22}$$

Let us note that the operator $\mathrm{sign}(\cdot)$ is a multivalued operator. In Appendix Section A.8 we provide some properties of sign function used along this work.

Now, as part of the control design, we assume that the positive constant D used in 4.21, is such that Assumption **(A1)**, see (4.5), is satisfied. From the fact that $\theta\mathrm{sign}(p) = |p|$ for all $\theta \in \mathrm{sign}(p)$, it follows that

$$-\|v_x\|^2_{L^2(0,L)} - D\mathrm{sign}(v(t,L))v(t,L) + d(t)v(t,L) \leq 0,\ \forall t \geq 0. \tag{4.23}$$

Thus, the right-hand side of (4.20) is negative. Therefore, the solution y to the following differential inclusion

$$\begin{cases} v_t - v_{xx} = -\omega v, & (t,x) \in (0,\infty) \times (0,L), \\ v_x(t,0) = 0, & t \in (0,\infty), \\ v_x(t,L) + D\mathrm{sign}(v(t,L)) \ni d(t), & t \in (0,\infty), \\ v(0,x) = v_0(x), & x \in (0,L), \end{cases} \tag{4.24}$$

satisfies that for any $\omega > 0$,

$$\|v(t,\cdot)\|^2_{L^2(0,L)} \leq e^{-2\omega t}\|v_0\|^2_{L^2(0,L)},\ \forall t \geq 0. \tag{4.25}$$

Summarizing, by using the transformation (4.13), the feedback control u in variable z is given by

$$u(t,z) = -k(L,L)z(t,L) - \int_0^L k_x(L,s)z(t,s)\,\mathrm{d}s \\ - D\mathrm{sign}\left(z(t,L) + \int_0^L k(L,s)z(t,s)\,\mathrm{d}s\right). \quad (4.26)$$

where $k = k(x,s)$ is solution to (4.14) and D is a positive constant such that satisfies Assumption **(A1)**, on Page 63.

4.3 Proof of Theorem 4.1.1

This section is dedicated to prove the main result of this work. That is, Theorem 4.1.1, which ensures, under suitable assumptions, the well-posedness of the system (4.1) in closed loop with the feedback (4.26), and the exponential decay in L^2 norm of its solution.

The system (4.1) in closed loop with the feedback law (4.26) is given by

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty) \times (0,L), \\ z_x(t,0) = 0, & t \in [0,\infty), \\ z_x(t,L) + k(L,L)z(t,L) + \int\limits_0^L k_x(L,s)z(t,s)\,\mathrm{d}s & \\ +D\mathrm{sign}\left(z(t,L) - \int\limits_0^L k(L,s)z(t,s)\,\mathrm{d}s\right) \ni d(t), & t \in [0,\infty), \\ z(0,x) = z_0(x), & x \in [0,L], \end{cases} \quad (4.27)$$

where $k = k(x,s)$ is solution to the gain kernel equation (4.3) and $D > 0$ is such that Assumption **(A1)**, see (4.5), is satisfied.

Let us notice that transformation (4.13) maps the closed-loop system (4.27) into the system

$$\begin{cases} v_t - v_{xx} = -\omega v, & (t,x) \in (0,\infty) \times (0,L), \\ v_x(t,0) = 0, & t \in (0,\infty), \\ v_x(t,L) + D\mathrm{sign}(v(t,L)) \ni d(t), & t \in (0,\infty), \\ v(0,x) = v_0(x), & x \in (0,L), \end{cases} \quad (4.28)$$

where $\omega > 0$.

Now, consider the following change of variable

$$w(t,x) = v(t,x) - \phi(x)d(t) \quad (4.29)$$

where $\phi : [0,L] \to \mathbb{R}$ is a function smooth enough such that $\phi'(0) = \phi(L) = 0$ and $\phi'(L) = 1$. For instance, $\phi(x) = \frac{1}{2L}x^2 - \frac{L}{2}$, for all x in $[0,L]$.

Then, w satisfies the following differential inclusion

$$\begin{cases} w_t - w_{xx} = -\omega w + f, & (t,x) \in (0,\infty) \times (0,L), \\ w_x(t,0) = 0, & t \in [0,\infty), \\ w_x(t,L) + D\mathrm{sign}(w(t,L)) \ni 0, & t \in [0,\infty), \\ w(0,x) = w_0(x), & x \in [0,L], \end{cases} \quad (4.30)$$

where $f = -\phi\dot{d} + \phi''d + \omega\phi d$. We recall that $\text{sign}(\cdot)$ is the multivalued operator given by (4.22).

In order to perform our analysis, let us introduce the following operator

$$\begin{aligned} \mathcal{A} : D(\mathcal{A}) \subset L^2(0,L) &\longrightarrow L^2(0,L), \\ p &\longmapsto \mathcal{A}p = -p'' + \omega p, \end{aligned} \tag{4.31}$$

where $\omega > 0$ and the domain $D(\mathcal{A})$ is given by

$$D(\mathcal{A}) = \left\{ p \in L^2(0,L)/\mathcal{A}p \in L^2 : p'(0) = 0,\ p'(L) + Dsign(p(L)) \ni 0 \right\}. \tag{4.32}$$

Let us notice that the operator $\mathcal{A}$ is not linear. In fact, the set $D(\mathcal{A})$ is not a linear subspace, in consequence the operator $\mathcal{A}$ can not be linear.

Now, the differential inclusion (4.30) can be written in an operator form as follows,

$$\begin{cases} w_t + \mathcal{A}w = f, & t \in (0,\infty), \\ w(0) = w_0. \end{cases} \tag{4.33}$$

In order to state the well-posedness of (4.33), we begin by proving the following proposition.

Proposition 4.3.1. *The operator $\mathcal{A}$ is a maximal monotone operator.*

Proof. By Minty's Theorem, see [40, Chapter III, Theorem 5], the operator $\mathcal{A}$ is a maximal monotone operator if and only if the operator $\mathcal{A}$ is monotone and the operator $\mathcal{I} + \mathcal{A}$ satisfies that $R(\mathcal{I} + \mathcal{A}) = L^2(0,L)$. We have denoted the identity operator as $\mathcal{I}$.

1. $\mathcal{A}$ is monotone. That is, for all $u, v \in D(\mathcal{A})$, $(\mathcal{A}u - \mathcal{A}v, u - v)_{L^2(0,L)} \geq 0$.

   Let $u, v \in D(\mathcal{A})$, then

$$\begin{aligned} (\mathcal{A}u - \mathcal{A}v, u - v)_{L^2(0,L)} = &\int_0^L (u' - v')^2 + \omega(u - v)^2 \,\mathrm{d}x \\ &- \big(u'(L) - v'(L)\big)\left(u(L) - v(L)\right). \end{aligned} \tag{4.34}$$

   Since that $u, v \in D(\mathcal{A})$, there exist $\tilde{u} \in sign(u(L))$ and $\tilde{v} \in sign(v(L))$ such that $u'(L) + D\tilde{u} = 0$ and $v'(L) + D\tilde{v} = 0$. This implies that

$$u'(L) - v'(L) = -D\left(\tilde{u} - \tilde{v}\right). \tag{4.35}$$

   Thus, replacing (4.35) into (4.34), we get

$$\begin{aligned} (\mathcal{A}u - \mathcal{A}v, u - v)_{L^2(0,L)} = &\int_0^L (u' - v')^2 + \omega(u - v)^2 \,\mathrm{d}x \\ &+ D\left(\tilde{u} - \tilde{v}\right)\left(u(L) - v(L)\right). \end{aligned} \tag{4.36}$$

   Since the operator $\text{sign}(\cdot)$ is a maximal monotone operator, see Proposition A.8.1 in Appendix section, last term in (4.36) is positive. Therefore, the operator $\mathcal{A}$ is monotone.

2. The operator $\mathcal{I} + \mathcal{A}$ has full rank.

Let us prove that $R(I + \mathcal{A}) = L^2(0, L)$, or in an equivalent form, for all $f \in L^2(0, L)$, there exists $p \in D(\mathcal{A})$ such that $p + \mathcal{A}p = f$ in $L^2(0, L)$.

Let us consider the Hilbert space $H^1(0, L)$ with its usual norm and the following functional, $\mathcal{J} : H^1(0, L) \to \mathbb{R}$, defined by

$$\mathcal{J}(p) = \frac{1}{2}\int_0^L (p')^2 + (\omega + 1)p^2 - fp \,\mathrm{d}x + \varphi_\lambda(p(L)), \tag{4.37}$$

where $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ is the Moreau Regularization, see [75, Chapter IV, Proposition 1.8] or see [71, Section 3.5.4], of the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi = D|x|$.

Let us consider $\alpha(x) = (\partial\varphi)(x) = D\mathrm{sign}(x)$, $J_\lambda(x) = (I + \lambda\alpha)^{-1}$, where J_λ is called the resolvent of α. Besides, we consider the Yosida approximation of α given by $\alpha_\lambda : \mathbb{R} \to \mathbb{R}$, $\alpha_\lambda(x) = \frac{1}{\lambda}(I - J_\lambda(x))$. See [75, Chapter IV, eq. (1.6)].

By the Moreau Theorem, see for instance [75, Chapter IV, Proposition 1.8], φ_λ is a convex, differentiable function and

$$\varphi_\lambda(x) = \frac{\lambda}{2}|\alpha_\lambda|^2 + \varphi(J_\lambda(x)), \quad \varphi_\lambda'(x) = \alpha_\lambda(x). \tag{4.38}$$

We need now the following lemma whose proof is given in Section A.9 in the Appendix.

Lemma 4.3.2. *For all $\lambda > 0$ and for all $f \in L^2(0, L)$, there exists a minimizer m_λ of $\mathcal{J}$ such that $m_\lambda \in H^2(0, L)$,*

$$\begin{cases} m_\lambda + \mathcal{A}m_\lambda = f, \quad a.e \quad x \in (0, L) \\ m_\lambda'(0) = 0, \quad m_\lambda'(L) + \alpha_\lambda(m_\lambda(L)) = 0. \end{cases} \tag{4.39}$$

Moreover, the minimizer m_λ satisfies the following inequalities. There exists positive constants C_i, $i \in \{1, 2, 3\}$ such that, for any $\lambda > 0$

$$\|m_\lambda\|_{H^1(0,L)} \le C_1\|f\|_{L^2(0,L)}, \tag{4.40}$$
$$\|m_\lambda\|_{H^2(0,L)} \le C_3\|f\|_{L^2(0,L)}, \tag{4.41}$$
$$|\alpha_\lambda(m_\lambda(L))| \le C_2\|f\|_{L^2(0,L)}. \tag{4.42}$$

We are in position to prove that $R(I+\mathcal{A}) = L^2(0, L)$. To do that, let us consider m_λ given by Lemma 4.3.2 and to analyze what happens with the sequences given by $\{m_\lambda\}_{\lambda>0}$ and $\{\alpha_\lambda(m_\lambda(L))\}_{\lambda>0}$, when $\lambda \to 0^+$.

Since both sequences are bounded (uniformly), there exists $(m, h) \in H^2(0, L) \times \mathbb{R}$ and subsequences, for the sake of simplicity we use the same notation, such that

$$m_\lambda \rightharpoonup m \in H^2(0, L), \quad \lambda \to 0^+, \tag{4.43}$$
$$\alpha_\lambda(m_\lambda(L)) \to h \in \mathbb{R}, \quad \lambda \to 0^+. \tag{4.44}$$

Here, we have used $\rightharpoonup$ to denote weak convergence.

In virtue of the compact injection of $H^2(0, L)$ into $C^1([0, L])$, [7, Theorem 8.8], we see that, $(m_\lambda)_{\lambda>0}$ converges strongly to m in $C^1([0, L])$. That is, $m_\lambda' \to m'$ and $m_\lambda \to m$ uniformly in $[0, L]$, as $\lambda \to 0^+$.

From Lemma 4.3.2, it holds that for every direction $r \in H^1(0,L)$, $\mathcal{J}'(m_\lambda; r) = 0$. In other words,

$$\int_0^L m_\lambda' r' + (\omega+1)m_\lambda r - fr\,\mathrm{d}x + \alpha_\lambda(m_\lambda(L))r(L) = 0, \quad \forall r \in H^1(0,L). \tag{4.45}$$

Let us notice that we have used that $\varphi_\lambda'(x) = \alpha_\lambda(x)$, see (4.38). Now, let us consider $r \in C_c^\infty([0,L]) \subset H^1(0,L)$ and take the limit as $\lambda \to 0^+$, to obtain

$$\lim_{\lambda\to 0^+} \int_0^L m_\lambda' r' + (\omega+1)m_\lambda r - fr\,\mathrm{d}x = \int_0^L m'r' + (\omega+1)mr - fr\,\mathrm{d}x = 0. \tag{4.46}$$

Performing one integration by parts, we obtain that

$$m + \mathcal{A}m = f, \quad \text{almost everywhere } x \in (0,L). \tag{4.47}$$

Now, by the uniformly convergence of m_λ' , we get that

$$\lim_{\lambda\to 0^+} m_\lambda'(0) = m'(0) = 0 \tag{4.48}$$

and by (4.44), we see that

$$\lim_{\lambda\to 0^+} m_\lambda'(L) + \alpha_\lambda(m_\lambda(L)) = m'(L) + h = 0. \tag{4.49}$$

In order to complete the proof, let us show that

$$h \in \alpha(m(L)). \tag{4.50}$$

We recall that $\alpha(m(L)) = D\mathrm{sign}(m(L))$ and that $\mathrm{sign}(\cdot)$ is a maximal monotone operator. Then α_λ, the Yosida approximation of α, satisfies that $\alpha_\lambda(x) \in \alpha(J_\lambda(x))$ for all $x \in \mathbb{R}$, where J_λ is the resolvent operator of α.

Thus, we have that for every $\lambda > 0$, $\alpha_\lambda(m_\lambda(L)) \in \alpha(J_\lambda(m_\lambda(L)))$. Then, since $\alpha_\lambda(m_\lambda(L)) \to h$, as $\lambda \to 0^+$ and that α is a closed operator, see [5, Proposition 2.1], it is sufficient to prove that $J_\lambda(m_\lambda(L)) \to m(L)$ as $\lambda \to 0^+$ to obtain (4.50).

To begin with, we recall the fact that the resolvent of a maximal monotone operator is Lipchitz continuous with constant equal to one, that is

$$|J_\lambda(x_1) - J_\lambda(x_2)| \le |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}. \tag{4.51}$$

Now, it holds that

$$\begin{aligned} |J_\lambda(m_\lambda(L)) - m(L)| &= |J_\lambda(m_\lambda(L)) - J_\lambda(m(L)) + J_\lambda(m(L)) - m(L)| \\ &\le |m_\lambda(L) - m(L)| + |J_\lambda(m(L)) - m(L)|. \end{aligned} \tag{4.52}$$

Then, from (4.52), and from the fact that $J_\lambda(x) \to x$, for all $x \in \mathbb{R}$, as $\lambda \to 0^+$ it follows that $J_\lambda(m_\lambda(L)) \to m(L)$, as $\lambda \to 0^+$. Thus, we see that $(m_\lambda)_{\lambda>0}$

converges to m which is solution of

$$\begin{cases} m + \mathcal{A}m = f, \quad a.e \quad x \in (0, L), \\ m'(0) = 0, \quad m'(L) + \alpha(m(L)) \ni 0. \end{cases} \tag{4.53}$$

Then, we conclude that $R(\mathcal{I} + \mathcal{A}) = L^2(0, L)$.

Thus, the operator $\mathcal{A}$ is maximal monotone. The proof of Proposition 4.3.1 is complete. □

Let us continue with the proof of the well-posedness of equation (4.33). Let assume Assumption **(A2)**, see (4.6) on page 63.

Thus, $f \in W^{1,1}(0, \infty; L^2(0, L))$. Since $\mathcal{A}$ is a maximal monotone operator and thanks to Theorem 4.1 in [75, Chapter 4], it holds that for any $w_0 \in D(\mathcal{A})$ there exists a unique $w \in W^{1,1}(0, \infty; L^2(0, L))$ solution to (4.33).

Now, from the regularity of w, it holds that $w, w_t \in L^1(0, \infty; L^2(0, L))$. Then, using the equation we get that $w \in W^{1,1}(0, \infty; L^2(0, L)) \cap L^1(0, \infty; H^2(0, L))$.

From the change of variable (4.29), and the assumption that $d(0) = 0$, we conclude that for any $v_0 \in D(\mathcal{A})$, there exists a unique solution v to (4.28), with $v \in W^{1,1}(0, \infty; L^2(0, L)) \cap L^1(0, \infty; H^2(0, L))$.

In order to conclude the well-posedness of the closed-loop system (4.27), let us consider the inverse transformation of the backstepping transformation (4.13), given by

$$z(x) = v(x) - \int_0^x l(x, s) v(s) \, \mathrm{d}s, \tag{4.54}$$

where $l = l(x, s)$ is a C^2 function on the triangle $\Omega = \left\{(x, s) \in \mathbb{R}^2 \ / \ 0 \le s \le x \le L\right\}$ being the unique solution to

$$\begin{cases} l_{xx}(x, s) - l_{ss}(x, s) = -(a(s) + \omega) l(x, s), & (x, s) \in \Omega, \\ l_s(x, 0) = 0, & x \in [0, L], \\ l(x, x) = \frac{1}{2} \int_0^x (a(s) + \omega) \, \mathrm{d}s, & x \in [0, L]. \end{cases} \tag{4.55}$$

The transformation (4.54) is a linear continuous transformation, see for instance [55, Lemma 3.3]. Thus, the set of admissible initial conditions for the closed-loop system (4.27), is the image of $D(\mathcal{A})$ under the transformation (4.54), which is equivalent to

$$\left\{z_0 \in H^2(0, L), \text{ such that } v_0'(0) = 0, \text{ and } v_0'(L) + D\mathrm{sign}(v_0(L)) \ni 0\right\}, \tag{4.56}$$

where we have defined

$$v_0 = z_0 + \int_0^x k(x, s) z_0(s) \, \mathrm{d}s, \tag{4.57}$$

with k solution to the gain kernel equation (4.3). The closed-loop system (4.27), has an unique solution $z \in W^{1,1}(0, \infty; L^2(0, L)) \cap L^1(0, \infty; H^2(0, L))$.

Finally the exponential decay of z follows from the continuity of the backstepping transformation (4.13) and its corresponding inverse (4.54). That is, there exists constants $C_1 > 1$, and $C_2 > 1$, such that

$$\|z(t,\cdot)\|^2_{L^2(0,L)} \leq C_1\|v(t,\cdot)\|^2_{L^2(0,L)} \leq C_1 e^{-2\omega t}\|v_0\|^2_{L^2(0,L)} \leq C_1C_2e^{-2\omega t}\|v_0\|^2_{L^2(0,L)} \tag{4.58}$$

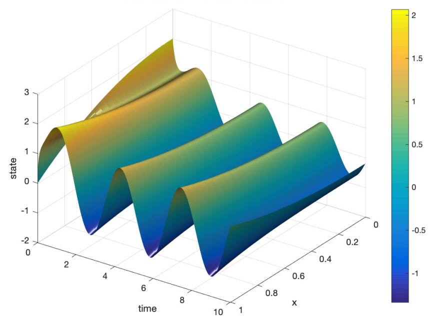
The proof of Theorem 4.1.1 is complete. □

4.4 Simulations

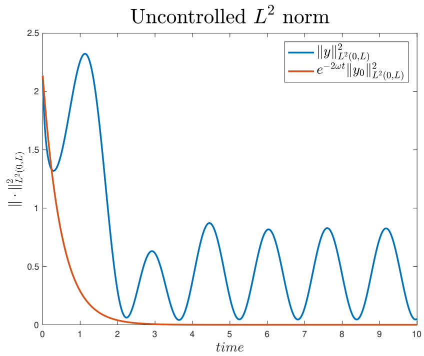
In this section we present some simulations in order to illustrate our theoretical stability result, stated in Theorem 4.1.1. To do that, we have discretized in space using finite difference centered method and solved the respective ODE system by using the explicit Euler method. By simplicity, we have simulated the control system (4.18).

In a first experiment, the parameters for simulations are the following, $L = 1$, $\omega = 1$, $D = 2$, and the initial condition $v_0(x) = 2(1-\frac{x^2}{2})$. Let us notice that v_0 satisfies the compatibility conditions, that is $v'(0) = 0$ and $v'(1) + 2\text{sign}(v(1)) = -2 + 2 = 0$. The disturbance signal is chosen as $d(t) = 2\sin(2t)$.

In Figure 4.1a, we have simulated the uncontrolled disturbed system, that is when $u_2(t) = 0$. Along with this, we have plotted the evolution of the correspond L^2 norm in Figure 4.1b. As we anticipated in the introduction, the disturbance signal is the source of instability for the system.



(a) Uncontrolled state



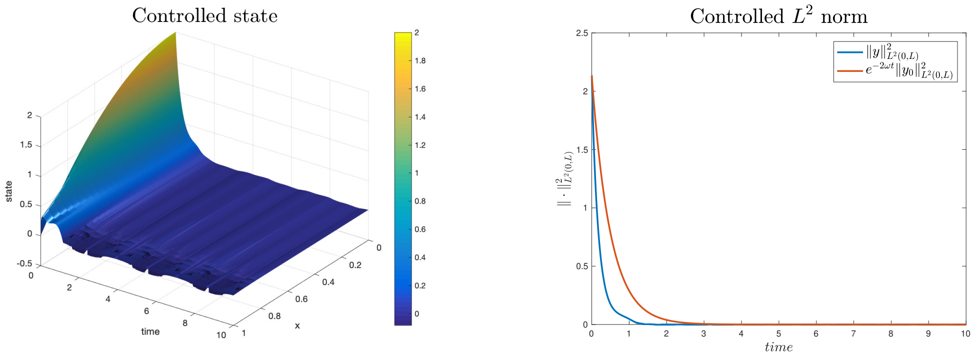
(b) Uncontrolled $L^2(0, L)$ norm

FIGURE 4.1: Uncontrolled state and its L^2 norm evolution.

Now, in Figure 4.2a we have simulated the closed-loop system (4.24), that is, when $u_2 = -D\text{sign}(v(t, L))$. Let us notice, it can be observed the action of the sign$(\cdot)$ operator, rejecting the disturbance and ensuring the exponential decay as is shown in Figure 4.2b.

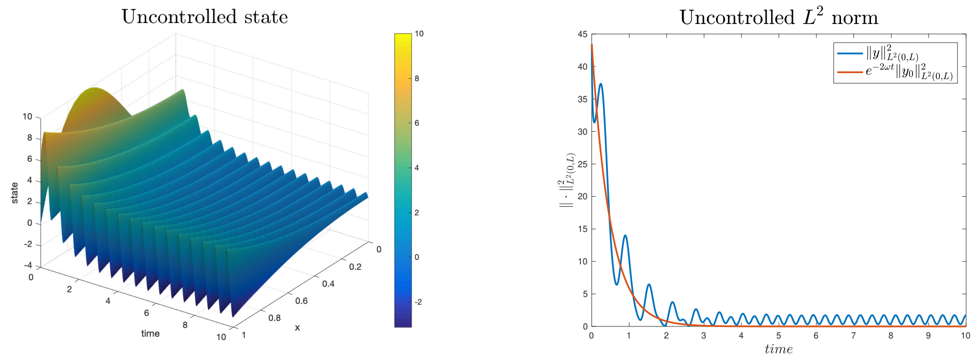
A second numerical experiment is plotted in Figure 4.3 and Figure 4.4. The parameter for simulations are the following $L = 1$, $\omega = 1$, $D = 10$. We have choose $v_0(x) = \frac{270}{4}(x^2 - x^3)$ as initial condition. We check that the compatibility conditions are satisfies, that is $v_0'(0) = 0$, $v_0'(1) + 10\text{sign}(v_0(1)) = -1 + [-10, 10] \ni 0$. The disturbance signal is given by $d(t) = 10\sin(10t)$.

In Figure 4.3a is plotted the uncontrolled state subject to disturbance signal and in Figure 4.3b is plotted the evolution of the L^2 norm. The same as before, see Figure 4.1, without the presence of the sign$(\cdot)$ to reject the disturbance, the system becomes unstable.



(a) Controlled state (b) Controlled $L^2(0, L)$ norm

FIGURE 4.2: Controlled state and its L^2 norm evolution.

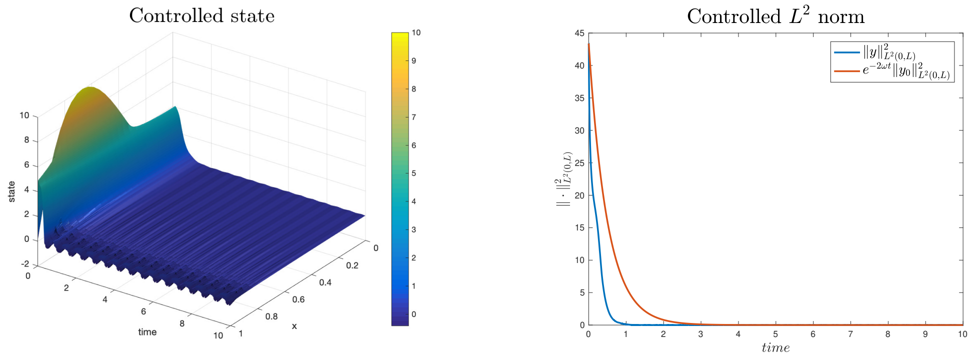


(a) Uncontrolled state (b) Uncontrolled norm $L^2(0, L)$ norm

FIGURE 4.3: Uncontrolled state and its L^2 norm evolution.

The Figure 4.4a shows the controlled state, and Figure 4.4 exhibit the exponential decay in L^2 norm, as we anticipated theoretically and confirmed in the previous experiment.

The main result of the this chapter establish the exponential decay in L^2 norm for the state, in consequence it is not possible to guarantee the exponential decay of the traces of the state at the boundaries, namely $v(t, 0)$ or $v(t, L)$ as it can be observed in Figure 4.2a and Figure 4.4a.



(a) Controlled state

(b) Controlled norm $L^2(0,L)$ norm

FIGURE 4.4: Controlled state and its L^2 norm evolution.

Chapter 5

Conclusions and perspectives

In this chapter we summarize the contributions and we give some remarks for every problem addressed in the previous chapters.

1. Along Chapter 2, the tracking problem of the State of Charge (SOC) to a given reference trajectory has been studied. The Single Particle Model ([12, 62]), which belongs to the class of Electrochemical models describing the dynamic of lithium ions concentration has been used. This tracking problem consisted in designing a current input for the battery such that the SOC converges to a prescribed trajectory as time goes to infinity.

   The approach to solve the tracking problem consisted, in a first stage, in designing an input feedback $I(t)$ which depends on the full ion concentration in the anode. The exponential convergence to zero of the tracking error has been proven. Moreover, an observer has been designed to avoid the online measurement of the full anode concentration. The proof was based on the backstepping method, yielding an exponential decay rate of the reference reference error for the SOC.

   An implicit difficult of this approach is that the ion concentration observer depends on an online boundary measurement of the lithium ion concentration. Even if only the boundary is measured, it is very difficult to get proper measurements. To avoid this difficult an observer has been designed for the ion concentration depending on an estimator of surface concentration satisfying Assumption 2.1.2. Some numerical simulations illustrated the obtained results.

   Possible future extensions naturally appear. We could consider models including the dynamics of the ions in electrolyte phase or a distributed temperature. Concerning the controller, a nice extension would be to consider saturated inputs.

2. In Chapter 3, the boundary null controllability, by the action of one single control, of two types of parabolic-elliptic systems has been proven.

   In a first case, we dealt with a parabolic-elliptic system with a non-linear term in the parabolic part of the system and with a control located on the boundary of the parabolic equation. As usual in that kind of problems, we begun studying the boundary null controllability of the linearized system around zero. To do that, the controllability observability duality principle has been used. Then, the observability inequality has been proven by considering the adjoint equation. In order to obtain such an inequality, the main tool was a Carleman estimate with boundary observation. Then, by means of a local inversion theorem, we proved a local null controllability result for the non-linear system.

   The second case consists of a linear system with constant coefficients and one single control placed at the boundary of the elliptic part. To state the control result, the strategy was different as before. Here we used a problem of moment approach and the spectral analysis of the underlying spatial operator associated to the system.

One possible future line research is to deal with linear parabolic-elliptic system with the control placed in the elliptic part and with non-constant coefficients. Here, the main difficult for a controllability-observability duality approach is to obtain the following observability inequality:

There exists a positive constant C, such that

$$\int_0^L w^2(0,x)\,\mathrm{d}x \leq C\int_0^T\int_0^L \theta^2(x)w^2(t,x)\,\mathrm{d}x\,\mathrm{d}t, \tag{5.1}$$

for w solution to (3.111), and θ solution to (3.108).

3. Finally, in Chapter 4, the rapid stabilization problem for an unstable heat equation under boundary disturbance has been addressed.

   To do that, it has been designed a control with two parts clearly defined. The first part, based on the backstepping method to deal with the instability caused by the positive part of the a coefficient, is designed in the same way as for the undisturbed case. See for instance [55]. The second part of the design proposed, is based on the use of the multivaluate operator $\operatorname{sign}(\cdot)$, defined in (4.22).

   The resulting system in closed loop is a differential inclusion. Thus, by means of the theory of monotone maximal operators, the well-posedness of the closed-loop system is stated.

   Numerical simulations were performed in order to illustrate the theoretical results obtained.

   One natural extension could be to consider variable coefficients in the main equation, also consider Dirichlet boundary conditions. Another interesting questions is about the same problem in higher dimension. Note that a difficulty comes from the lack of the backstepping method for dimensions greater than 1.

   Another interesting open question is the following.

   Let us note that the initial conditions for which Theorem 4.1.1 is valid might not be vast. Then, is it possible to use a density argument in order to introduce the notion of mild solutions to (4.1)? With the purpose to extend the results of Theorem 4.1.1, in particular (4.11), to any initial condition in $L^2(0,L)$, a natural conjecture is the following

   Conjecture 5.1. *Let us consider the hypotheses of Theorem 4.1.1 except the ones of the initial condition. Let $z_0 \in L^2(0,L)$. Then, there exists a unique mild solution $z = z(t,x)$ in $C([0,\infty); L^2(0,L))$ to*

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty)\times(0,L),\\ z_x(t,0) = 0, & t \in (0,\infty),\\ z_x(t,L) \ni u(t,z) + d(t), & t\in(0,\infty),\\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{5.2}$$

   Moreover, (5.2) is exponentially stable in $L^2(0,L)$, with decay $\omega > 0$. In other words, there exists a positive constant $C > 1$, such that the solution z to the *closed-loop system* (5.2) *satisfies that*

$$\|z(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|z_0\|_{L^2(0,L)} \text{ for every } t \in [0,\infty). \tag{5.3}$$

The main question here is how to conclude the density in $L^2(0,L)$ of the set of admissible initial conditions in Conjecture 5.1. In other words, is the set

$$\left\{z_0 \in H^2(0,L), \text{ such that } v_0'(0) = 0,\ v_0'(L) + D\mathrm{sign}(v_0(L))) \ni 0\right\}, \tag{5.4}$$

where $v_0 = z_0 + \int_0^x k(x,s)z_0(s)\,\mathrm{d}s$, a dense set in $L^2(0,L)$?

Appendix A

Appendix

A.1 Proof of Proposition 2.2.1

Consider a given reference trajectory $SOC_{ref}(t)$. We look for an input $I(t)$ to regulate the $SOC(t)$ of the system (2.7) to $SOC_{ref}(t)$. To do that we define

$$\kappa(t) = \frac{1}{2}\left(SOC_{ref}(t) - SOC(t)\right)^2. \tag{A.1}$$

Now, taking the time derivative over $\kappa(t)$ and using the system (2.7) we get

$$\dot{\kappa}(t) = \left(SOC_{ref}(t) - SOC(t)\right)\left(S\dot{O}C_{ref}(t) - \int_0^1 c_t r^2 \,\mathrm{d}r\right), \tag{A.2}$$

$$= \left(SOC_{ref}(t) - SOC(t)\right)\left(S\dot{O}C_{ref}(t) - \frac{3\tilde{\rho}}{c_{\max}} I(t)\right). \tag{A.3}$$

Then, if we select the current as

$$I(t) = \frac{c_{\max}}{3\tilde{\rho}}\left(S\dot{O}C_{ref}(t) + \gamma\left(SOC_{ref}(t) - SOC(t)\right)\right), \tag{A.4}$$

where $\gamma > 0$ is a constant design parameter, then we obtain

$$\dot{\kappa}(t) = -\gamma\left(SOC_{ref}(t) - SOC(t)\right)^2, \tag{A.5}$$

$$= -2\gamma\kappa(t). \tag{A.6}$$

Last equation implies that $\kappa(t) = \kappa(0)e^{-2\gamma t}$ and in particular

$$\lim_{t\to\infty} \kappa(t) = 0. \tag{A.7}$$

In conclusion we have that $|SOC_{ref}(t) - SOC(t)| \to 0$ when $t \to \infty$. $\square$

A.2 Proof of Proposition 2.3.3

Consider the linear operator $A : D(A) \subset L_r^2(0,1) \to L_r^2(0,1)$ defined by $A\tilde{z} := -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tilde{z}_r)$ and $D(A) = \{\tilde{z} \in H_r^2(0,1) : \tilde{z}_r(0) = \tilde{z}_r(1) = 0\}$. It is easy to check that A is maximal monotone. Thus, by the Hille-Yosida theorem (see Theorem 7.4 in [7, Chapter 7]), if $z_0 \in D(A)$, then equation (2.25) has a unique solution $\tilde{z} \in C([0,\infty); D(A)) \cap C^1([0,\infty); L_r^2(0,1))$.

Now, we perform some energy estimations. For the moment we assume $z_0 \in D(A)$ and then we easily obtain that the solutions to (2.25) satisfy

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{z}\|^2_{L^2_r(0,1)} = -\lambda\int_0^1 \widetilde{z}^2 r^2\,\mathrm{d}r - \int_0^1 \widetilde{z}_r^2 r^2\,\mathrm{d}r \le -\lambda\|\widetilde{z}\|^2_{L^2_r(0,1)} \tag{A.8}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{z}_r\|^2_{L^2_r(0,1)} = -\lambda\int_0^1 \widetilde{z}_r^2 r^2 dr - \int_0^1 (\widetilde{z}_{rr}r + 2\widetilde{z}_r)^2\,\mathrm{d}r. \tag{A.9}$$

Consequently, we get the inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\widetilde{z}\|^2_{L^2_r(0,1)} + \|\widetilde{z}_r\|^2_{L^2_r(0,1)}\right) \le -2\lambda\left(\|\widetilde{z}\|^2_{L^2_r(0,1)} + \|\widetilde{z}_r\|^2_{L^2_r(0,1)}\right)&\\ - 2\|\widetilde{z}_{rr}r + 2\widetilde{z}_r\|^2_{L^2(0,1)}.&\end{aligned} \tag{A.10}$$

and applying the Gronwall's lemma we get

$$\|\widetilde{z}(t,\cdot)\|^2_{L^2_r(0,1)} + \|\widetilde{z}_r(t,\cdot)\|^2_{L^2_r(0,1)} \le e^{-2\lambda t}\left(\|\widetilde{z}(0,\cdot)\|^2_{L^2_r(0,1)} + \|\widetilde{z}_r(0,\cdot)\|^2_{L^2_r(0,1)}\right) \tag{A.11}$$

given (2.26). This inequality also allows to use a density argument to conclude that (2.25) has a unique solution $\widetilde{z} \in C([0,\infty); H^1_r(0,1)) \cap C^1([0,\infty); L^2_r(0,1))$ if $\widetilde{z}_0 \in H^1_r(0,1)$. □

A.3 Proof of Lemma 2.3.4

Consider the following function

$$\check{p}(r,s) = \frac{r}{s}p(r,s). \tag{A.12}$$

After some calculations we get that

$$p_r(r,s) = -\frac{s}{r^2}\check{p} + \frac{s}{r^2}\check{p}_r, \tag{A.13}$$

$$p_{rr}(r,s) = -\frac{2s}{r^3}\check{p} - \frac{s}{r^2}\check{p}_r - \frac{s}{r^2}\check{p}_r + \frac{s}{r}\check{p}_{rr}, \tag{A.14}$$

$$p_s(r,s) = \frac{1}{r}\check{p} + \frac{s}{r}\check{p}_s, \tag{A.15}$$

$$p_{ss}(r,s) = \frac{1}{r}\check{p}_s + \frac{1}{r}\check{p}_s + \frac{s}{r}\check{p}_s, \tag{A.16}$$

then, using (2.27) and equations (A.13)-(A.16), we get the following equation and boundary conditions for $\check{p}(r,s)$

$$\begin{cases}\check{p}_{rr}(r,s) - \check{p}_{ss}(r,s) = -\lambda\check{p}(r,s), & (r,s)\in T,\\ \check{p}(r,0) = 0, \quad \check{p}(r,r) = -\frac{\lambda}{2}r, & r\in(0,1).\end{cases} \tag{A.17}$$

Using the Successive Approximations Method we solve the equation (A.17), see [49, Chapter 4], and we obtain that

$$\check{p}(r,s) = -\lambda r\frac{J_1\left(\sqrt{\lambda(r^2-s^2)}\right)}{\left(\sqrt{\lambda(r^2-s^2)}\right)}, \tag{A.18}$$

where J_1 is the first order Bessel function of first kind. Then the kernel $p(r,s)$ is given by (2.30). Equation (2.11) follows from (2.30). This concludes the proof of Lemma 2.3.4. □

Remark A.3.1. *From the kernel transformation* (A.12) *and boundary condition of* (2.27) *we observe that the boundary condition* $\check{p}(r,0)$ *remains free. The selection of* $\check{p}(r,0)=0$ *ensures a well-posed equation* (A.17) *and an explicit solution.*

A.4 Proof of Theorem 2.3.8

Consider the quadratic tracking error

$$\kappa(t)=\frac{1}{2}\left(SOC_{ref}(t)-SOC(t)\right)^2,$$

and take the time derivative. Thus, we obtain

$$\begin{aligned}\dot{\kappa}(t)&=\left(SOC_{ref}(t)-SOC(t)\right)\left(S\dot{O}C_{ref}(t)-\frac{3\tilde{\rho}}{c_{\max}}I(t)\right)\\&=-\gamma\left(SOC_{ref}(t)-SOC(t)\right)\left(SOC_{ref}(t)-\widehat{SOC}(t)\right)\\&=-\gamma\left(SOC_{ref}(t)-SOC(t)\right)\left(SOC_{ref}(t)-SOC(t)+SOC(t)-\widehat{SOC}(t)\right)\\&=-\gamma\left(SOC_{ref}(t)-SOC(t)\right)^2-\gamma\left(SOC_{ref}(t)-SOC(t)\right)\left(SOC(t)-\widehat{SOC}(t)\right).\end{aligned}$$

Moreover, by the Young inequality, for all $t\geq 0$,

$$\begin{aligned}&\left|-\gamma\left(SOC_{ref}(t)-SOC(t)\right)\left(SOC(t)-\widehat{SOC}(t)\right)\right|\leq\\&\qquad\frac{\gamma}{2}\left(SOC_{ref}(t)-SOC(t)\right)^2+\frac{\gamma}{2}\left(SOC(t)-\widehat{SOC}(t)\right)^2.\end{aligned}\tag{A.19}$$

Then we obtain that for all $t\geq 0$,

$$\dot{\kappa}(t)\leq-\gamma\kappa(t)+\frac{\gamma}{2}\left(SOC(t)-\widehat{SOC}(t)\right)^2.\tag{A.20}$$

By Proposition 2.3.6, we obtain that for all $t\geq 0$,

$$\dot{\kappa}(t)+\gamma\kappa(t)\leq\frac{3\gamma M^2}{2c_{\max}^2}\|\widetilde{c}_0\|^2_{H_r^1}e^{-2\lambda t}.\tag{A.21}$$

Multiplying by $e^{\gamma t}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\kappa(t)e^{\gamma t}\right)\leq\frac{3\gamma M^2}{2c_{\max}^2}\|\widetilde{c}_0\|^2_{H_r^1}e^{(\gamma-2\lambda)t}\tag{A.22}$$

From the above inequality we distinguish three cases depending on the value of γ.

1. Let $\gamma<2\lambda$. Integrating (A.22) over $(0,t)$ we get

$$\kappa(t)e^{\gamma t}-\kappa(0)\leq\frac{3\gamma M^2}{2c_{\max}^2}\frac{\|\widetilde{c}_0\|^2_{H_r^1}}{(\lambda-2\gamma)}(e^{(\gamma-2\lambda)t}-1).\tag{A.23}$$

We ignore the negative terms in the righthand side of the previous inequality to get, for all $t \geq 0$

$$\kappa(t) \leq \left(\kappa^2(0) + \frac{3\gamma M^2}{2c_{\max}^2} \frac{\|\widetilde{c}_0\|_{H_r^1}^2}{|\gamma - 2\lambda|} \right) e^{-\gamma t}. \tag{A.24}$$

2. Let $\gamma = 2\lambda$. We integrating (A.22) over $(0,t)$, we get for all $t \geq 0$

$$\kappa(t) \leq \left(\kappa^2(0) + \frac{3\gamma M^2}{2c_{\max}^2} \|\widetilde{c}_0\|_{H_r^1}^2 t \right) e^{-\gamma t}. \tag{A.25}$$

3. Let $\gamma > 2\lambda$. Similar as before, we integrate (A.22) over $(0,t)$ to get for all $t \geq 0$

$$\kappa(t) \leq \kappa^2(0) e^{-\gamma t} + \frac{3\gamma M^2}{2c_{\max}^2} \frac{\|\tilde{c}_0\|_{H_r^1}^2}{(\gamma - 2\lambda)} e^{-2\lambda t}. \tag{A.26}$$

Finally we collect the inequalities (A.24), (A.25) and (A.26) to conclude. The proof of Theorem 2.3.8 is complete. □

A.5 Proof of Lemma 3.1.1

The operator F_γ is well defined and linear continuous. Indeed, it follows from the fact that (3.4) is a well posed equation for every $\gamma \in L^\infty(0,L)$ such that $\gamma(x) > \gamma_0 \geq -\pi^2/L^2$ for all $x \in [0,L]$ and for all $g \in L^2(0,L)$, see for instance, [7, Chapter 8] or [74, Chapter 8].

Let us show that F_γ is self-adjoint. On the one hand, consider $\phi \in L^2(0,L)$, by definition of F_γ it holds

$$\int_0^L F_\gamma(g)\phi \,\mathrm{d}x = \int_0^L \zeta\phi \,\mathrm{d}x, \quad \forall \phi \in L^2(0,L), \tag{A.27}$$

where ζ is a solution to (3.4). On the other hand, multiplying (3.4) by $h \in C_c^\infty(0,L)$ and performing an integration by parts over $[0,L]$, we get

$$\int_0^L \zeta(-h_{xx} + \gamma(x)h) \,\mathrm{d}x = \int_0^L gh \,\mathrm{d}x \tag{A.28}$$

Letting $\phi = -h_{xx} + \gamma(x)h$, we get $h = F_\gamma(\phi)$. This allows us to conclude that

$$\int_0^L F_\gamma(g)\phi = \int_0^L \zeta\phi dx = \int_0^L gh \,\mathrm{d}x = \int_0^L gF_\gamma(\phi) \,\mathrm{d}x. \tag{A.29}$$

Then F_γ is a self adjoint operator.

Now, we estimate a continuity constant for F_γ. Multiply (3.4) by its solution ζ and then perform an integration by parts on the left-hand side, we get

$$\int_0^L (\zeta_x^2 + \gamma\zeta^2) \,\mathrm{d}x = \int_0^L g\zeta \,\mathrm{d}x, \quad \forall g \in L^2(0,L). \tag{A.30}$$

Using the Cauchy-Schwartz inequality on the right-hand side, we get

$$\int_0^L (\zeta_x^2 + \gamma\zeta^2)\,\mathrm{d}x \leq \|g\|_{L^2(0,L)}\|\zeta\|_{L^2(0,L)}, \quad \forall g \in L^2(0,L). \tag{A.31}$$

First, suppose that $\gamma_0 \geq 0$. By the Poincaré inequality, $\|v\|_{L^2(0,L)} \leq \frac{L}{\pi}\|v_x\|_{L^2(0,L)}$ for all $v \in H_0^1(0,L)$, we get from (A.31)

$$\|\zeta_x\|_{L^2(0,L)} \leq \frac{L}{\pi}\|g\|_{L^2(0,L)}, \quad \forall g \in L^2(0,L). \tag{A.32}$$

Now, suppose that $-\pi^2/L^2 < \gamma_0 < 0$. Then, taking account (A.31) and in virtue of the Poincaré inequality, it holds

$$\left(1+\gamma_0\frac{L^2}{\pi^2}\right)\int_0^L \zeta_x^2\,\mathrm{d}x \leq \int_0^L (\zeta_x^2 + \gamma_0\zeta^2)\,\mathrm{d}x \leq \frac{L}{\pi}\|g\|_{L^2(0,L)}\|\zeta_x\|_{L^2(0,L)}, \tag{A.33}$$

which implies that

$$\|\zeta_x\|_{L^2(0,L)} \leq \frac{L\pi}{\pi^2 + \gamma_0 L^2}\|g\|_{L^2(0,L)}, \quad \forall g \in L^2(0,L). \tag{A.34}$$

Thus, we get

$$\|\zeta\|_{H_0^1(0,L)} \leq C(\gamma_0, L)\|g\|_{L^2(0,L)}, \quad \forall g \in L^2(0,L), \tag{A.35}$$

where $C(\gamma_0, L) = \max\left\{\frac{L}{\pi}, \frac{L\pi}{\pi^2+\gamma_0 L^2}\right\}$. The proof of Lemma 3.1.1 is complete. □

A.6 Proof of Proposition 3.3.1

Pick $w_T \in H_0^1(0,L)$ and $h \in L^1(0,T,H_0^1(0,L))$, let be w solution of the adjoint equation (3.66) with final data w_T. Consider the following differential operator,

$$Pw = -w_t - w_{xx}. \tag{A.36}$$

Besides, we recall the weight function, given by

$$\varphi(t,x) := \frac{\beta(x)}{t(T-t)}, \quad \beta(x) = -\left(\frac{x}{L} - 2\right)^2 + 8, \quad (t,x) \in [0,T]\times[0,L]. \tag{A.37}$$

Let $\lambda > 0$ and ψ defined by (A.37). Let us set the following change of variables

$$v = e^{-\lambda\varphi}w, \quad P_\varphi v = e^{-\lambda\varphi}P(e^{\lambda\varphi}v). \tag{A.38}$$

Thus, we can write $P_\varphi v = P_1 v + P_2 v + Rv$, where

$$\begin{aligned} P_1 v &= -v_t - 2\lambda\varphi_x v_x - 2\lambda\varphi_{xx}v, \\ P_2 v &= -v_{xx} - \lambda^2\varphi_x^2 v - \lambda\varphi_t v, \\ Rv &= \lambda\varphi_{xx}v. \end{aligned} \tag{A.39}$$

Taking the L^2-norm in $\Omega = [0,T]\times[0,L]$ to $P_1v + P_2v = P_\varphi v - Rv$, it holds

$$\|P_1v\|^2_{L^2(\Omega)} + 2(P_1v, P_2v)_{L^2(\Omega)} + \|P_2v\|^2_{L^2(\Omega)} = \|P_\varphi v - Rv\|^2_{L^2(\Omega)}. \tag{A.40}$$

From which it follows that

$$(P_1v, P_2v)_{L^2(\Omega)} \leq \|P_\varphi v\|^2_{L^2(\Omega)} + \|Rv\|^2_{L^2(\Omega)}. \tag{A.41}$$

Now, we bound from below the inner product $(P_1v, P_2v)_{L^2(\Omega)}$.

To begin with, let us computing $(P_1v, P_2v)_{L^2(\Omega)}$. We denote I_{ij}, $(i,j)\in\{1,2,3\}^2$, the L^2-product in Ω between the i-th term of P_1v and j-th term of P_2v. Integrations by parts are performed. In order to keep a simple notation we omit the limits on the double integrals and the symbol $\mathrm{d}x\,\mathrm{d}t$.

$$I_{11} = \int_0^T v_t v\big|_0^L \,\mathrm{d}t - \frac{1}{2}\iint \frac{\mathrm{d}}{\mathrm{d}t}v_x^2, \tag{A.42}$$

$$I_{12} = -\lambda^2 \iint \varphi_{xt}\varphi_x v^2, \tag{A.43}$$

$$I_{13} = -\frac{\lambda}{2}\iint \varphi_{tt}v^2, \tag{A.44}$$

$$I_{21} = \lambda\int_0^T \varphi_x v_x^2\big|_0^L \,\mathrm{d}t - \lambda\iint \varphi_{xx}v_x^2, \tag{A.45}$$

$$I_{22} = \lambda^3\int_0^T \varphi_x^3 v^2\big|_0^L \,\mathrm{d}t - 3\lambda^3\iint \varphi_x^2\varphi_{xx}v^2, \tag{A.46}$$

$$I_{23} = \lambda^2\int_0^T \varphi_t\varphi_x v^2\big|_0^L \,\mathrm{d}t - \lambda^2\iint \varphi_{xt}\varphi_x v^2 - \lambda^2\iint \varphi_t\varphi_{xx}v^2, \tag{A.47}$$

$$I_{31} = -\lambda\int_0^T \varphi_{xxx}v^2\big|_0^L \,\mathrm{d}t + 2\lambda\int_0^T \varphi_{xx}vv_x\big|_0^L \,\mathrm{d}t + \lambda\iint \varphi_{xxxx}v^2 - 2\lambda\iint \varphi_{xx}v_x^2, \tag{A.48}$$

$$I_{32} = 2\lambda^3\iint \varphi_x^2\varphi_{xx}v^2, \tag{A.49}$$

$$I_{33} = 2\lambda^2\iint \varphi_t\varphi_{xx}v^2. \tag{A.50}$$

Several terms in (A.42)-(A.50) can be reduced to zero using that $v = e^{-\lambda\varphi}w$ and that $\varphi\to\infty$ if $t\to 0$ or $t\to T$. Besides, the boundary conditions of the adjoint equation (3.66) implies that $v(0) = v(L) = 0$. Also note that $\partial_x^k\varphi = 0$ if $k > 2$. Thus, we can define the following decomposition of the inner product $(P_1v, P_2v)_{L^2(\Omega)}$ as follows,

$$(P_1v, P_2v)_{L^2(\Omega)} = D(v) + B(v) \tag{A.51}$$

where the distributed and boundary terms are given by

$$D(v) = -\lambda^3\iint \varphi_x^2\varphi_{xx}v^2 - 3\lambda\iint \varphi_{xx}v_x^2 - 2\lambda^2\iint \varphi_{xt}\varphi_x v^2$$

$$+\lambda^2 \iint \varphi_t \varphi_{xx} v^2 - \frac{\lambda}{2} \iint \varphi_{tt} v^2. \quad \text{(A.52)}$$

$$B(v) = \lambda \int_0^T \varphi_x v_x^2 \big|_0^L \, \mathrm{d}t \quad \text{(A.53)}$$

Now, using the weight function defined in (A.37) we will bound from below the distributed terms $D(w)$. To do that, we use the following bounds for φ and its derivatives

1. $\dfrac{16}{T^2} \leq \varphi(t,x), \ \forall \ (t,x), \ [0,T] \times [0,L]$,
2. $\dfrac{2}{7L}\varphi(t,x) \leq \varphi_x(t,x) \leq \dfrac{1}{L}\varphi(t,x), \ \forall \ (t,x), \ [0,T] \times [0,L]$,
3. $\dfrac{2}{7L^2}\varphi(t,x) \leq |\varphi_{xx}(t,x)| \leq \dfrac{1}{2L^2}\varphi(t,x), \ \forall \ (t,x), \ [0,T] \times [0,L]$,
4. $\varphi_t(t,x) \leq \dfrac{T}{4}\varphi^2(t,x), \ \forall \ (t,x), \ [0,T] \times [0,L]$,
5. $\varphi_{xt}(t,x) \leq \dfrac{T}{L}\varphi^2(t,x), \ \forall \ (t,x), \ [0,T] \times [0,L]$,
6. $\varphi_{tt}(t,x) \leq \dfrac{T^2}{8}\varphi^3(t,x), \ \forall \ (t,x), \ [0,T] \times [0,L]$.

Consider the two first term in (A.52). Note that $\varphi_{xx} < 0$ and using the inequalities listed above, it holds

$$-\lambda^3 \iint \varphi_x^2 \varphi_{xx} v^2 - 3\lambda \iint \varphi_{xx} v_x^2 \geq \frac{8}{343L^4}\lambda^3 \iint \varphi^3 v^2 + \frac{6}{7L^2}\lambda \iint \varphi v_x^2. \quad \text{(A.54)}$$

These leading terms will allow to us to absorb the remaining terms in (A.52) and the residual term Rv by choosing $\lambda > 0$ in a suitable way.

The following inequalities can be obtained by using bounds for the weight function φ.

- $\left| 2\lambda^2 \iint \varphi_x \varphi_{xt} v^2 \right| \leq \dfrac{2T}{L^2\lambda} \iint \lambda^3 \varphi^3 v^2.$
- $\left| \lambda^2 \iint \varphi_{xx} \varphi_t v^2 \right| \leq \dfrac{T}{8L^2\lambda} \iint \lambda^3 \varphi^3 v^2.$
- $\left| \dfrac{\lambda}{2} \iint \varphi_{tt} v^2 \right| \leq \dfrac{T^2}{16\lambda^2} \iint \lambda^3 \varphi^3 v^2.$

Using (A.54) and the previous inequalities we bound from below (A.52), as follows

$$D(v) \geq \left(\frac{8}{343L^4} - \frac{17T}{8L^2}\frac{1}{\lambda} - \frac{T^2}{16}\frac{1}{\lambda^2} \right) \iint \lambda^3 \varphi^3 v^2 + \frac{6}{7L^2} \iint \lambda \varphi v_x^2. \quad \text{(A.55)}$$

Now, we are going to bound form above the L^2 norm of the residual term Rv

$$\|Rv\|^2_{L^2(\Omega)} \leq \lambda^2 \iint \varphi_{xx}^2 v^2 \leq \frac{T^2}{64L^4}\frac{1}{\lambda} \iint \lambda^3 \varphi^3 v^2. \quad \text{(A.56)}$$

Then, collecting (A.41), (A.51), (A.55) and (A.56), we get

$$\left(\frac{8}{343L^4} - \frac{17T}{8L^2}\frac{1}{\lambda} - \frac{T^2}{16}\frac{1}{\lambda^2}\right)\iint \lambda^3\varphi^3 v^2 + \frac{6}{7L^2}\iint \lambda\varphi v_x^2 + \lambda\int_0^T \varphi_x v_x^2\Big|_0^L \mathrm{d}t$$
$$\leq \|P_\varphi v\|^2_{L^2(\Omega)} + \frac{T^2}{16L^4}\frac{1}{\lambda}\iint \lambda^3\varphi^3 v^2. \quad \text{(A.57)}$$

Rearranging terms, and taking account that $\varphi_x(L,t) \geq 0$ for any $t \in (0,T)$, it holds

$$D_0(\lambda)\iint \lambda^3\varphi^3 v^2 + D_1\iint \lambda\varphi v_x^2 \leq \|P_\varphi v\|^2_{L^2(\Omega)} + \lambda\int_0^T \varphi_x(t,0)v_x^2(t,0)\,\mathrm{d}t, \quad \text{(A.58)}$$

where

$$D_0(\lambda) = \left(\frac{8}{343L^4} - \frac{136TL^2+T^2}{64L^4}\frac{1}{\lambda} - \frac{T^2}{16}\frac{1}{\lambda^2}\right), \quad \text{(A.59)}$$
$$D_1 = \frac{6}{7L^2}. \quad \text{(A.60)}$$

In order to handle (A.59), we consider $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ it holds

$$D_0(\lambda) \geq \frac{4}{343L^4} \quad \text{(A.61)}$$

Let us define $D_2 = \frac{4}{343L^4}$, then it holds for all $\lambda \geq \lambda_0$

$$D_2\iint \lambda^3\varphi^3 v^2 + D_1\iint \lambda\varphi v_x^2 \leq \|P_\varphi v\|^2_{L^2(\Omega)} + \lambda\int_0^T \varphi_x(t,0)v_x^2(t,0)\,\mathrm{d}t. \quad \text{(A.62)}$$

Now, we define $\frac{1}{C_3} = \min\{D_2, D_1\}$. From the previous inequality, we can conclude that there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$

$$\lambda^3\iint \varphi^3 v^2 + \lambda\iint \varphi v_x^2 \leq C_3\left(\|P_\varphi v\|^2_{L^2(\Omega)} + \lambda\int_0^T \varphi_x(t,0)v_x^2(t,0)\,\mathrm{d}t\right). \quad \text{(A.63)}$$

Finally, to obtain the Carleman estimate (3.68) note that from the change of variable $w = e^{\lambda\varphi}v$ it holds

$$\lambda^3\iint e^{-2\lambda\varphi}\varphi^3 w^2 + \lambda\iint e^{-2\lambda\varphi}\varphi w_x^2 = \lambda^3\iint e^{-2\lambda\varphi}\varphi^3\left(e^{\lambda\varphi}v\right)^2 + \lambda\iint e^{-2\lambda\varphi}\varphi\left(e^{\lambda\varphi}v\right)_x^2. \quad \text{(A.64)}$$

Developing $(e^{\lambda\varphi})_x$ and having in mind the bounds of the φ, it is not difficult to prove that there exists a constant $C_4 > 0$ such that

$$\lambda^3\iint e^{-2\lambda\varphi}\varphi^3\left(e^{\lambda\varphi}v\right)^2 + \lambda\iint e^{-2\lambda\varphi}\varphi\left(e^{\lambda\varphi}v\right)_x^2 \leq C_4\left(\lambda^3\iint \varphi^3 v^2 + \lambda\iint \varphi v_x^2\right), \quad \text{(A.65)}$$

for λ large enough. Then, using (A.63) and recalling the variable change defined by (A.38) we conclude that there exists a constant $C_5 > 0$ such that

$$\begin{aligned}&\lambda^3 \iint e^{-2\lambda\varphi}\varphi^3 w^2 + \lambda \iint e^{-2\lambda\varphi}\varphi w_x^2\\ &\le C_5 \left(\iint e^{-2\lambda\varphi}(F_\gamma(w) - (q+f'(0))w + h)^2 + \lambda \int_0^T \varphi_x(t,0)(e^{-\lambda\varphi(t,x)}w)_x^2(t,0)\,\mathrm{d}t \right),\end{aligned} \tag{A.66}$$

for λ large enough. Developing the trace term $(e^{-\lambda\varphi(t,x)}w)_x^2(t,0)$ we get (3.68). The proof of Proposition 3.3.1 is complete. □

A.7 Proof of Lemma 3.3.13

On one hand,

$$\prod_{k=1,k\neq m}^{\infty} \frac{\lambda_m + \lambda_k}{|\lambda_m + \lambda_k|} = \exp\left(\sum_{k=1,k\neq m}^{\infty} \ln\left(\frac{\lambda_m + \lambda_k}{|\lambda_m + \lambda_k|} \right) \right). \tag{A.67}$$

On the other hand, since that $\lambda_k \ge 0$ for any $k \in \mathbb{N}$ and thanks to the triangle inequality, we see that

$$\lambda_k \le |\lambda_m - \lambda_k| + \lambda_m \tag{A.68}$$

then,

$$\frac{\lambda_m + \lambda_k}{|\lambda_m - \lambda_k|} \le 1 + \frac{2\lambda_m}{|\lambda_m - \lambda_k|}. \tag{A.69}$$

Thefore, combining (A.67) and (A.69) we get that,

$$\exp\left(\sum_{k=1,k\neq m}^{\infty} \ln\left(\frac{\lambda_m + \lambda_k}{|\lambda_m + \lambda_k|} \right) \right) \le \exp\left(\sum_{k=1,k\neq m}^{\infty} \ln\left(1 + \frac{2\lambda_m}{|\lambda_m - \lambda_k|} \right) \right). \tag{A.70}$$

Now, we can compare

$$\sum_{k=1,k\neq m}^{\infty} \log\left(1 + \frac{2\lambda_m}{|\lambda_m - \lambda_k|} \right) \le \int_1^{\infty} \ln\left(1 + \frac{2\lambda_m}{\left| \lambda_m - q_0 - \left(\frac{\pi}{L}\right)^2 x^2 + \frac{1}{\left(\frac{\pi}{L}\right)^2 x^2 + \gamma_0} \right|} \right) \mathrm{d}x, \tag{A.71}$$

For simplicity we are going to assume that $\gamma_0 > 0$, when $\gamma_0 = 0$ the analysis is similar. Then

$$\int_1^{\infty} \ln\left(1 + \frac{2\lambda_m}{\left| \lambda_m - q_0 - \left(\frac{\pi}{L}\right)^2 x^2 + \frac{1}{\left(\frac{\pi}{L}\right)^2 x^2 + \gamma_0} \right|} \right) \mathrm{d}x \le$$

$$\int_0^\infty \ln\left(1+\frac{2\lambda_m}{\left|\lambda_m - q_0 - \left(\frac{\pi}{L}\right)^2 x^2 + \frac{1}{\left(\frac{\pi}{L}\right)^2 x^2+\gamma_0}\right|}\right) \mathrm{d}x. \quad \text{(A.72)}$$

Let us define

$$g(x) = \lambda_m - q_0 - \left(\frac{\pi}{L}\right)^2 x^2 + \frac{1}{\left(\frac{\pi}{L}\right)^2 x^2 + \gamma_0}. \quad \text{(A.73)}$$

Since that $\lambda_m - q_0 \geq \left(\frac{m\pi}{L}\right)^2 - \frac{1}{\left(\frac{\pi}{L}\right)^2+\gamma_0} \geq 0$, let us consider $\hat{x} = \sqrt{(\lambda_m - q_0)}/\frac{\pi}{L}$.

The function g is continuous and decrescent in x and $g(\hat{x}) > 0$. Then, there exists a unique $\bar{x}$ such that $g(\bar{x}) = 0$. Let us consider this $\bar{x} = \delta\sqrt{(\lambda_m - q_0)}/\frac{\pi}{L}$. Since g is monotone, then $\delta > 1$.

Thus,

$$\begin{aligned}\int_0^\infty \ln\left(1+\frac{2\lambda_m}{|g(x)|}\right) \mathrm{d}x &= \int_0^{\sqrt{\lambda_m-q_0}/\frac{\pi}{L}} \ln\left(1+\frac{2\lambda_m}{|g(x)|}\right) \mathrm{d}x + \int_{\sqrt{\lambda_m-q_0}/\frac{\pi}{L}}^{\delta\sqrt{\lambda_m-q_0}/\frac{\pi}{L}} \ln\left(1+\frac{2\lambda_m}{|g(x)|}\right) \mathrm{d}x \\ &+ \int_{\delta\sqrt{\lambda_m-q_0}/\frac{\pi}{L}}^{(\delta+1)\sqrt{\lambda_m-q_0}/\frac{\pi}{L}} \ln\left(1+\frac{2\lambda_m}{|g(x)|}\right) \mathrm{d}x + \int_{(\delta+1)\sqrt{\lambda_m-q_0}/\frac{\pi}{L}}^{\infty} \ln\left(1+\frac{2\lambda_m}{|g(x)|}\right) \mathrm{d}x \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad \text{(A.74)}$$

In the following we study each one of these integrals.

- **Estimation of I_1**

  Since $x \in \left(0, \sqrt{\lambda_m - q_0}/\frac{\pi}{L}\right)$, then $|g(x)| = g(x)$. Moreover,

$$\frac{1}{g(x)} \leq \frac{1}{\lambda_m - q_0 - \left(\frac{\pi}{L}\right)^2 x^2} \quad \text{(A.75)}$$

  Then

$$I_1 \leq \int_0^{\sqrt{\lambda_m-q_0}/\frac{\pi}{L}} \ln\left(1+\frac{2\lambda_m}{\lambda_m - q_0 - \left(\frac{\pi}{L}\right)^2 x^2}\right) \mathrm{d}x \quad \text{(A.76)}$$

  Let us consider the change of variable $x = \frac{\sqrt{\lambda_m - q_0}}{\pi/L} z$, then

$$I_1 \leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} \int_0^1 \ln\left(1+\frac{2\lambda_m}{(\lambda_m - q_0)(1-z^2)}\right) dz \quad \text{(A.77)}$$

  On one hand, note that for $m \geq 1$

$$\frac{\lambda_m}{\lambda_m - q_0} = 1 + \frac{q_0}{\lambda_m - q_0}, \quad \text{(A.78)}$$

  then $\frac{\lambda_m}{\lambda_m - q_0} \leq \frac{\lambda_1}{\lambda_1 - q_0} = c$ for all $m \geq 1$.

On the other hand, for any $z \in (0,1)$ it holds that $1/(1-z^2) \leq 1/(1-z)$. Then, it can be conclude that

$$I_1 \leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} \int_0^1 \ln\left(1 + \frac{c}{1-z}\right) dz. \tag{A.79}$$

In order to compute the previous integral, we consider that

$$\int_0^1 \ln\left(1 + \frac{c}{1-z}\right) dz = -\int_0^1 (1-z)' \ln\left(1 + \frac{c}{1-z}\right) dz. \tag{A.80}$$

By integrations by parts, get that

$$\int_0^1 (1-z)' \ln\left(1 + \frac{c}{1-z}\right) dz =$$

$$-(1-z)\ln\left(1 + \frac{c}{1-z}\right)\Bigg|_0^1 - c\int_0^1 \frac{1}{(c+1)-z} dz = \omega_1 \leq \infty \tag{A.81}$$

Finally, we get that

$$I_1 \leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L}\omega_1. \tag{A.82}$$

- **Estimation of I_2**

Since $x \in (\sqrt{\lambda_m - q_0}/\frac{\pi}{L}, \delta\sqrt{\lambda_m - q_0}/\frac{\pi}{L})$, then $|g(x)| = g(x)$ and recalling that $g(x)$ is decreasing in x then

$$\frac{1}{g(x)} \leq \frac{1}{\lambda_m - q_0 + \frac{1}{\delta^2(\lambda_m - q_0) + \gamma_0} - \left(\frac{\pi}{L}\right)^2 x^2}. \tag{A.83}$$

Thus,

$$\int_{\sqrt{\lambda_m - q_0}/\frac{\pi}{L}}^{\delta\sqrt{\lambda_m - q_0}/\frac{\pi}{L}} \ln\left(1 + \frac{2\lambda_m}{g(x)}\right) \mathrm{d}x$$

$$\leq \int_{\sqrt{\lambda_m - q_0}/\frac{\pi}{L}}^{\delta\sqrt{\lambda_m - q_0}/\frac{\pi}{L}} \ln\left(1 + \frac{2\lambda_m}{\lambda_m - q_0 + \frac{1}{\delta^2(\lambda_m - q_0) + \gamma_0} - \left(\frac{\pi}{L}\right)^2 x^2}\right) \mathrm{d}x. \tag{A.84}$$

Consider now, the change of variable $x = \frac{\sqrt{\lambda_m - q_0}}{\pi/L} z$, then

$$\int_{\sqrt{\lambda_m - q_0}/\frac{\pi}{L}}^{\delta\sqrt{\lambda_m - q_0}/\frac{\pi}{L}} \ln\left(1 + \frac{2\lambda_m}{\lambda_m - q_0 + \frac{1}{\delta^2(\lambda_m - q_0) + \gamma_0} - \left(\frac{\pi}{L}\right)^2 x^2}\right) \mathrm{d}x$$

$$\leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} \int_1^{\delta} \ln\left(1 + \frac{\frac{2\lambda_m}{\lambda_m - q_0}}{1 + \frac{1/(\lambda_m - q_0)}{\delta^2(\lambda_m - q_0)+\gamma_0} - z^2}\right) dz$$

$$\leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} \int_1^{\delta} \ln\left(1 + \frac{c}{1 - z^2}\right) dz \quad \text{(A.85)}$$

Now, recalling that $\frac{1}{1-z^2} = \frac{1}{1-z}\frac{1}{1+z} \leq \frac{2}{1-z}$ for $z \in (1, \delta)$, then it can be conclude that

$$I_2 \leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} \int_1^{\delta} \ln\left(1 + \frac{2c}{1 - z}\right) dz \leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} w_2 < \infty \quad \text{(A.86)}$$

- **Estimation of I_3.**
  Since $x \in (\delta\sqrt{\lambda_m - q_0}/\frac{\pi}{L}, (\delta+1)\sqrt{\lambda_m - q_0}/\frac{\pi}{L})$, $|g(x)| = -g(x)$. Then,

$$0 \leq \frac{1}{-g(x)} \leq \frac{1}{\left(\frac{\pi}{L}\right)^2 x^2 - (\lambda_m - q_0) - \frac{1}{\gamma_0 + \delta^2(\lambda_m - q_0)}} \quad \text{(A.87)}$$

Then, using the change of variable $x = \frac{\sqrt{\lambda_m - q_0}}{\pi/L} z$, we see that

$$I_3 \leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} \int_{\delta}^{\delta+1} \ln\left(1 + \frac{c}{z^2 - b_m}\right) dz$$

$$\leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} \int_{\delta}^{\delta+1} \ln\left(1 + \frac{c}{(z - \sqrt{b_m})^2}\right) dz, \quad \text{(A.88)}$$

where $b_m = \left(1 + \frac{1/(\lambda_m - q_0)}{\delta^2(\lambda_m - q_0)+\gamma_0}\right)$.
To compute the previous integral we consider that

$$\int_{\delta}^{\delta+1} \ln\left(1 + \frac{c}{(z - \sqrt{b_m})^2}\right) dz = \int_{\delta}^{\delta+1} (z - \sqrt{b_m})' \ln\left(1 + \frac{c}{(z - \sqrt{b_m})^2}\right) dz. \quad \text{(A.89)}$$

Then, by integration by parts we get that

$$\int_{\delta}^{\delta+1} (z - \sqrt{b_m})' \ln\left(1 + \frac{c}{(z - \sqrt{b_m})^2}\right) dz =$$

$$(z - \sqrt{b_m}) \ln\left(1 + \frac{c}{(z - \sqrt{b_m})^2}\right)\Bigg|_{\delta}^{\delta+1} + \int_{\delta}^{\delta+1} \frac{2c}{c + (z - \sqrt{b_m})^2} dz$$

$$\leq w_3 < \infty. \quad \text{(A.90)}$$

Then,

$$I_3 \leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L}\omega_3. \tag{A.91}$$

- **Estimation of I_4**

  Since $x \in ((\delta+1)\sqrt{\lambda_m - q_0}/\frac{\pi}{L}, \infty)$, $|g(x)| = -g(x)$. Similar as before, to obtain the estimation for I_3, it holds that

$$\begin{aligned} I_4 \leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} \int_{\delta}^{\delta+1} \ln\left(1 + \frac{c}{(z-\sqrt{b_m})^2}\right) dz &\leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L} \int_{\delta}^{\delta+1} \frac{c}{(z-\sqrt{b_m})^2} dz \\ &\leq \frac{\sqrt{\lambda_m - q_0}}{\pi/L}\omega_4 < \infty \end{aligned} \tag{A.92}$$

Now, collecting (A.67), (A.71), (A.82), (A.86), (A.91) and (A.92) we get that, there exist positive constance M and ω such that

$$\prod_{k=1,k\neq m}^{\infty} \frac{\lambda_m + \lambda_k}{|\lambda_m + \lambda_k|} = M \exp\left(\omega\sqrt{\lambda_m}\right). \tag{A.93}$$

The proof of Lemma 3.3.13 is complete. □

A.8 Properties of multivalued operator $sign(\cdot)$

Proposition A.8.1. *The multivalued operator $sign(\cdot)$ defined by*

$$\begin{aligned} sign &: \mathbb{R} \to 2^{\mathbb{R}} \\ f \mapsto sign(f) &= \begin{cases} \frac{f}{|f|} & if \quad f \neq 0, \\ [-1,1] & if \quad f = 0, \end{cases} \end{aligned} \tag{A.94}$$

where $2^{\mathbb{R}}$ denotes the power set of $\mathbb{R}$, is a maximal monotone operator.

Proof. Let us check that operator $\mathrm{sign}(\cdot)$ is monotone. That is, for $(x_1, y_1), (x_2, y_2) \in Graph(\mathrm{sign}(\cdot))$ it holds that $(y_2 - y_1)\cdot(x_2 - x_1) \geq 0$. Let assume that $x_1, x_2 \neq 0$, the other cases follows analogous.

$$\begin{aligned} (y_2 - y_1)\cdot(x_2 - x_1) &= \left(\frac{x_2}{|x_2|} - \frac{x_1}{|x_1|}\right)\cdot(x_2 - x_1), \\ &= |x_1| + |x_2| - x_1 x_2\left(\frac{1}{|x_2|} + \frac{1}{|x_1|}\right), \\ &\geq 0. \end{aligned}$$

Last inequality follows from the fact that, $|x_1 x_2| \leq |x_1|\,|x_2|$.

Now, from Figure A.1 we can observe directly, that the graph of $\mathrm{sign}(\cdot)$ is not properly contained in the graph of any other monotone operator. Thus, $\mathrm{sign}(\cdot)$ a maximal monotone operator. The proof of Proposition A.8.1 is complete. □

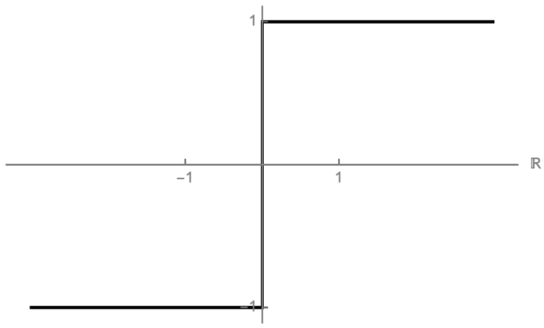


FIGURE A.1: Multivalued operator $\text{sign}(\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$.

A.9 Proof of Lemma 4.3.2

We recall the functional considered in Lemma 4.3.2, which is given by $\mathcal{J} : H^1(0, L) \to \mathbb{R}$, defined by

$$\mathcal{J}(p) = \frac{1}{2} \int_0^L (p')^2 + (\omega + 1)p^2 - fp \, \mathrm{d}x + \varphi_\lambda(p(L)), \tag{A.95}$$

where $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ is the Moreau Regularization, see [75, Chapter IV, Proposition 1.8] of the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi = D|x|$.

Besides, let us set $\alpha(x) = (\partial\varphi)(x) = D\text{sign}(x)$. $J_\lambda(x) = (I + \lambda\alpha(x))^{-1}$, where J_λ is called the resolvent of α and let us consider the Yosida approximation of α given by $\alpha_\lambda : \mathbb{R} \to \mathbb{R}$, $\alpha_\lambda(x) = \frac{1}{\lambda}(I - J_\lambda(x))$. See [75, Chapter IV, eq. (1.6)].

Now, by the Moreau Theorem, see for instance [75, Chapter IV, Proposition 1.8], φ_λ is a convex, differentiable function and

$$\varphi_\lambda(x) = \frac{\lambda}{2}|\alpha_\lambda|^2 + \varphi(J_\lambda(x)), \quad \varphi_\lambda'(x) = \alpha_\lambda(x). \tag{A.96}$$

The functional $\mathcal{J}$, is well-defined, convex, continuous and coercive on $H^1(0, L)$. In fact, the continuity of $\mathcal{J}$ follows from the continuous injection from $H^1(0, L)$ into $C(0, L)$ and since that φ_λ is a non-negative term in (A.95), see (A.96), it holds

$$\mathcal{J}(p) \geq \min\left\{\frac{1}{2}, \frac{\omega + 1}{2}\right\} \|p\|^2_{H^1(0,L)} - \|f\|_{L^2(0,L)}\|p\|_{H^1(0,L)}, \ \forall p \in H^1(0, L). \tag{A.97}$$

which implies the coercivity of the functional $\mathcal{J}$ on $H^1(0, L)$. The functional $\mathcal{J}$ has at least one minimizer, this in virtue of [71, Theorem 2.19]. Let m be a minimizer of $\mathcal{J}$, then the directional derivative of $\mathcal{J}$ at m vanish in every direction $r \in H^1(0, L)$, that is

$$\mathcal{J}'(m; r) = \int_0^L m'r' + (\omega + 1)mr - fr \, \mathrm{d}x + \alpha_\lambda(m(L))r(L) = 0, \ \forall r \in H^1(0, L). \tag{A.98}$$

Let $r \in C_c^\infty([0,L]) \subset H^1(0,L)$, we get

$$\int_0^L m' r' \,\mathrm{d}x = -\int_0^L ((\omega+1)m - f) r \,\mathrm{d}x, \ \forall r \in C_c^\infty([0,L]). \tag{A.99}$$

Since that $(\omega+1)m - f \in L^2(0,L)$, this implies that $m' \in H^1(0,L)$ and therefore $m \in H^2(0,L)$. Moreover, after one integration by parts in (A.99) it holds that

$$\int_0^L \big(-m'' + (\omega+1)m - f\big) \, r \,\mathrm{d}x = 0, \ \forall r \in C_c^\infty([0,L]). \tag{A.100}$$

which implies that $-m'' + (\omega+1)m - f = 0$ almost everywhere $x \in (0,L)$.

Now, performing one more integration by parts in (A.98), it holds that

$$m'(0)r(0) + \big(m'(L) + \alpha_\lambda(m(L))\big)\, r(L) = 0, \ \forall r \in H^1(0,L). \tag{A.101}$$

From which, we conclude that $m'(0) = 0$ and $m'(L) + \alpha_\lambda(m(L)) = 0$.

Finally, in order to get the inequalities, (4.40)-(4.42). Let $\lambda > 0$ and m_λ be a minimizer of $\mathcal{J}$, indexed by λ, fix $r = m_\lambda$ in (A.98), it holds that

$$\min\{1, \omega+1\} \left(\|m_\lambda\|^2_{L^2(0,L)} + \|m'_\lambda\|^2_{L^2(0,L)}\right) + m_\lambda(L)\alpha_\lambda(m_\lambda(L)) \leq \\ \|f\|_{L^2(0,L)} \|m_\lambda\|_{L^2(0,L)}. \tag{A.102}$$

We recall that α_λ is the Yosida of the maximal monotone operator $\operatorname{sign}(\cdot)$. Thus, α_λ is monotone as well, then $m_\lambda(L)\alpha_\lambda(m_\lambda(L)) \geq 0$.

This implies that there exists $C_1 > 0$, such that

$$\|m_\lambda\|_{H^1(0,L)} \leq C_1 \|f\|_{L^2(0,L)}. \tag{A.103}$$

That is inequality (4.40). Now, let us consider that

$$\|m''_\lambda\|^2_{L^2(0,L)} = \int_0^L m''_\lambda \left(f - (\omega+1)m_\lambda\right) \mathrm{d}x. \tag{A.104}$$

Then, applying Cauchy-Schwartz inequality on the right-hand side of the previous equality and using inequality (A.103), we obtain

$$\|m_\lambda\|_{H^2(0,L)} \leq C_2 \|f\|_{L^2(0,L)}. \tag{A.105}$$

Finally, let us note that $\alpha_\lambda(m_\lambda(L)) = -m'(L)$ and that

$$\max_{x \in [0,L]} \left|m'_\lambda(x)\right| \leq \|m'_\lambda\|_{H^1(0,L)}.$$

Then, in addition to inequality (A.105), it holds that there exists a constant $C_3 > 0$ such that

$$|\alpha_\lambda(m_\lambda(L))| \leq C_3 \|f\|_{L^2(0,L)}. \tag{A.106}$$

The proof of Lemma 4.3.2 is complete □

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