2016

ON CFT AND CONFORMAL TECHNIQUES IN AdS

RAMIREZ KRAUSE, ISRAEL ADOLFO

Universidad Técnica Federico Santa María

Departamento de Física

Valparaíso, Chile

On CFTs and conformal techniques in *AdS*

Israel Adolfo Ramírez Krause

A thesis presented for the degree of

Doctor of Science

November 2016

Doctoral Thesis

presented in

Departamento de Física

Universidad Técnica Federico Santa María

by

Israel Adolfo Ramírez Krause

To earn the degree in PhD in Physics

Title

On CFTs and conformal techniques in AdS

Directors

Brenno Carlini Vallilo and Alfonso Zerwekh

Valparaíso, Chile. November 2016

Dedicada a la Alegría de mis Ojos

Abstract

Conformal symmetry is an essential tool for the study of string theory, critical phenomena and interacting quantum field theories, among other examples. In this thesis, we focus on conformal techniques for two theories: Type IIB superstring theory on an $AdS_5 \times S^5$ background, and $\mathcal{N} = 2$ theories in four dimensions.

On the $\mathcal{N} = 2$ theory, we will give a first step towards the computation of superconformal blocks for mixed operators. For chiral and real half-BPS operators, their superconformal block expansion can be achieved using chiral or harmonics superspace techniques, respectively. For more general multiplets, no general tool is available. A first step towards this goal is to obtain the OPE for those general multiplets. In this thesis we show how to compute mixed OPEs between an $\mathcal{N} = 2$ stress-tensor multiplet, a chiral multiplet and a flavor current multiplet using superspace techniques. A general bound for the central charge of interacting theories will be obtained using the $\mathcal{N} = 2$ stress-tensor multiplet OPE.

On the string theory side, we propose a systematic way to compute the logarithmic divergences of composite operators in the pure spinor description of the $AdS_5 \times S^5$ superstring. The computations of these divergences can be summarized in terms of a dilatation operator acting on the local operators. We check our results with some important composite operators of the formalism. Finally, we construct the pure spinor AdS string using supertwistors.

Acknowledgements

I would like to express my gratitude to my advisors, Alfonso Zerwekh and Brenno Vallilo, for their patience, comprehension and guidance. In the same spirit, I want to express my deep gratitude towards Pedro Liendo, who helped me beyond all possible, if any, obligation. I am sure that his honesty, which sometimes seemed to be just too much honesty for one day, sharpened my skills and clarified my knowledge about this turmoil called academia. I also want to thank to Prof. Matthias Staudacher for his invitation to his group. He gave me a unique opportunity.

I wasn't born when I first laid my eyes upon Sofía Retamales, but it seems to be that a lifetime of happiness has passed since we've been together. I want to thank you, my Love, for always been there, in all kind of shape and form, for me and for us. The Sun does not rise until my gaze finds your smile.

I also want to thank to all the good friends I have made during this long journey, Giorgos "The Yogurth" Anastasiou, Sebastián "Doglover" Acevedo, Dario "Córtate Esa Chasca" López, Cesar "Hank" Arias, Per(dido) Sundell, Álvaro Seisdedos, Nelson "Bolita" Tabilo, Rodrigo Olea and Olivera Miskovic, here in Chile. We have shared great moments, a lot of fun, many interesting discussion and more than a few drinks. My gratitude also goes to my friends in Berlin, Christian "The Blond Terror" Marboe, Pavel "The Rockstar" Friedrich and his gang, Edoardo Vescovi and Vladimir Mitev. I learned from them that warm friends can melt any snow! I cannot forget the good times, falls and twisted joints acquired in all the Judo clubs (Samurai, Universidad Santa María and the Erste Berliner Judo Club) and BJJ clubs (José Luis Figueroa and Apu's group at the Erste Berliner Judo Club.)

I cannot forget my parents and my parents-in-law. Their tolerance and patience increases with every new meeting.

Finally, I wan to thank to María Loreto Vergara and Elizabeth Muga, who always helped me to unknot the Chilean bureaucracy.

This thesis was supported by the CONICYT scholarship, which could be a lot better if they cut the unnecessary and, more often than not, contradictory bureaucracy. I also received economical support from my home institution at different stages and with several different names.

Contents

Abstract **i**

Acknowledgements **ii**

1 Introduction **1**

1.1 Conformal Symmetry 1

1.2 The String Side 4

1.3 The CFT Side 7

1.4 Overview of this Thesis 8

2 Worldsheet dilatation operator for the AdS superstring **10**

2.1 Introduction 10

2.2 Renormalization of operators in the principal chiral model 13

2.3 Dilatation operator for the $AdS_5 \times S^5$ superstring 19

2.3.1 Pure spinor AdS string 19

2.3.2 General coset model coupled to ghosts 21

2.3.3 Pairing rules 26

2.4 Applications 27

2.4.1 Stress-energy tensor 27

2.4.2 Conserved currents 28

2.4.3 b ghost 30

2.5 Conclusions and further directions . . . 32

2.A Notation and conventions . . . 33

2.B Some identities for $\mathfrak{psu}(2,2|4)$ . . . 34

2.C Complete solution of the SD equation for the $AdS_5 \times S^5$ pure spinor string . 35

2.C.1 Action . . . 36

2.C.2 Schwinger-Dyson equation and the Interaction Matrix . . . 37

2.C.3 Green functions . . . 40

2.C.4 Pairing rules . . . 45

3 Supertwistor description of the AdS pure spinor string **51**

3.1 Introduction . . . 51

3.2 The $GL(4|4)/(GL(1) \times Sp(2))^2$ coset . . . 52

3.3 Symmetries of the AdS Space . . . 58

3.3.1 Local . . . 59

3.3.2 Global . . . 61

3.4 BRST transformation and BRST invariant action . . . 62

3.4.1 PSU-formalism . . . 63

3.4.2 GL-formalism . . . 65

3.4.3 Vectors . . . 68

3.5 An application: vertex operator construction . . . 71

3.5.1 Equation of Motions . . . 72

3.5.2 Construction of V . . . 73

3.6 Conclusion and further directions . . . 75

4 Stress-tensor OPE in $\mathcal{N} = 2$ superconformal theories **77**

4.1 Introduction . . . 77

4.2 Preliminaries . . . 80

4.3 Three-point functions . . . 84

4.3.1 Solutions 89

4.3.2 Extra solutions 92

4.4 2d chiral algebra and central charge bound 94

4.4.1 Enhanced Virasoro symmetry 97

4.5 Superconformal block analysis 100

4.5.1 Quick review of conformal blocks 100

4.5.2 Toward the superconformal block 101

4.A $\mathcal{N} = 2$ superconformal algebra 105

4.B Superspace identities 106

5 Mixed OPEs in $\mathcal{N} = 2$ superconformal theories **108**

5.1 Introduction 108

5.2 The three-point function of $\mathcal{N} = 2$ SCFT 110

5.3 Mixed OPE 113

5.3.1 $\mathcal{E}_q \times \hat{\mathcal{C}}_{0(0,0)}$ 115

5.3.2 $\mathcal{E}_q \times \hat{\mathcal{B}}_1$ 118

5.3.3 $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{B}}_1$ 120

5.4 Discussion 123

5.A Long, short and semi-short multiplets 124

5.B Discarded solutions 127

Chapter 1

Introduction

1.1 Conformal Symmetry

Conformal symmetry plays a major role in several areas of modern physics: it characterises phase transitions at critical points, such as liquid-vapour and superfluidity phase transitions; it is the full symmetry of the worldsheet generated by a relativistic string; it is also the symmetry of QCD at high energy and it provides an extension of the Poincaré algebra without conflicting with the Coleman-Mandula theorem, just to mention a few examples. It is this last property that makes conformal symmetry so appealing for the high energy physicist: as a general rule, the more symmetry a theory possesses, the simpler it is [1]. Thus, when Poincaré symmetry alone is not enough to solve a relativistic quantum field theory (QFT), allowing a larger symmetry group than the Poincaré group might be enough to solve the theory.

Indeed, this is the case in two dimensions, where the conformal symmetry group is described by an infinite dimensional Virasoro algebra [2]. This infinite amount of symmetry was a key point to completely solve and classify a family of two dimensional conformal field theories (CFTs) called two dimensional minimal models [3]. A key idea behind the solution of those models was the conformal bootstrap [4, 5, 6]. The conformal bootstrap makes use of the full conformal symmetry and consistency conditions, such as unitarity[1], crossing symmetry and closure of the operator product expansion (OPE)[2] to constrain the theory. In dimensions higher than two, the conformal group is finite dimensional.[3] Therefore, it is natural to ask whether the conformal symmetry is strong enough to solve a CFT in higher dimensions.

Recently, a huge amount of progress in this direction began since the work of [9] where, instead of trying to solve a CFT making use of its symmetry, the question the authors of [9] tried to answer was which conditions are consistent in the CFT. Imposing crossing symmetry, unitarity and closure of the OPE, they were able to numerically constrain the possible values of the three-point function coefficients. For a review see [8, 10, 11]. As it was latter discovered, such consistency conditions severely constrain the theory, imposing numerical bounds not only to the three-point function coefficient, but also to the conformal dimension of the possible operators in the CFT [12, 13, 14, 15, 16, 17].

There is another way to enlarge the Poincaré group: by extending its Lie algebra to a graded Lie algebra, the super-Poincaré algebra. In four dimensions, the simplest super-Poincaré algebra consists on the usual Poincaré algebra plus one spinorial generator, which generates supersymmetry (SUSY), a symmetry between fermionic and bosonic fields. It is possible to introduce more than one SUSY generator. A theory with $\mathcal{N}$ different supercharges is said to have an $\mathcal{N}$-extended SUSY. The most symmetric theory in four dimensions which only contains fields with spin no larger than one is $\mathcal{N} = 4$ Super Yang-Mills (SYM). This theory is, at the same time, conformal. This is why it is known as a superconformal field theory (SCFT). A remarkable conjecture relates the $\mathcal{N} = 4$ SYM theory, which does not contain gravity (which is mediated by a field of spin two), with a Type IIB superstring

[1] Although there are many interesting non-unitarity models, such as effective theories, in this thesis we will focus only on unitarity theories.

[2] This is a very important difference between a QFT and a CFT. Unlike QFTs, particles states cannot be defined in a CFT. A two-particle state in a QFT is a non-local operators that can be written as the product of two local (single particle) operators only locally. No such problem arises in a CFT, where the OPE is an exact relation. For more on this issue, see [7].

[3] Actually, the symmetry group of the conformal group in D dimensions, with $D > 2$ is the one corresponding to $D + 2$ Poincaré symmetry group $SO(D, 2)$ [8].

theory in ten dimensions living on a $AdS_5 \times S^5$[4] target space [18]. This conjecture is known as the AdS/CFT correspondence. This correspondence relates the strong (weak) coupling limit of a string theory to the weak (strong) coupling limit of $\mathcal{N} = 4$ SYM. This is the reason why it is called a duality. Other examples of dual theories are the Sine-Gordon, which is dual to the Thirring model, the Ising model which is dual to itself and, type IIB superstring is also dual to itself.

String theory started in the 60's as a theory of Hadrons which was able to reproduce the Veneziano amplitude from the scattering of four tachyons[5]. Instead of treating particles as points in space-time, string theory works with one-dimensional objects (strings.) These objects span a two-dimensional surface known as the worldsheet, in the same way as a point particle spans a one dimensional world-line. This two dimensional theory is, as all the theories discussed so far, a CFT. It was later discovered that string theory contains a graviton in its spectrum [19], giving the first consistent model of quantum gravity[6], the only force missing in the standard model of particle physics. The standard model successfully provides a unified framework for the electromagnetic, weak and strong interactions based on quantum field theory. One of its landmarks is the prediction of the Higgs boson, which was experimentally discovered in 2013. Despite all of its accomplishments, the Standard Model fails to include gravity. During the middle of the '80s, during a period dubbed *The First Superstring Revolution,* it was realized that superstring theory could provide a unified description of not only gravity, but of all particles and their interactions. Around ten years latter, in a period called the *Second Superstring Revolution*, not only perturbative effects were studied. S-Duality, T-Duality and Mirror symmetry started to play an important role, and all the five superstring theories were related through several dualities [20]. It was also discovered that string theory is a theory of not only strings, but also of extended objects of

[4] AdS_n is the maximally symmetric n-dimensional space with negative curvature. It has the same amount of symmetry as flat space (Minkowski) and dS_n space (which has positive curvature.) S^n is the sphere in $n+1$ dimensions.

[5] Latter, QCD was established as the correct theory for Hadrons. Despite of its success, it is still not known how the Regge trajectories emerge from QCD at long distances.

[6] So far, string theory is still the only consistent theory of quantum gravity.

various dimensions, known as branes [21]. One of the most important ideas of this period is the already mentioned AdS/CFT correspondence between a string theory which contains gravity, to $\mathcal{N} = 4$ SYM, a CFT with no gravity, living in the boundary of the AdS_5 space. Originally, this conjecture relates the weak (strong) coupling limit of a stack of N D3-branes which corresponds to a supergravity solution of an $AdS_5 \times S^5$ space-time with the strong (weak) coupling limit of a $SU(N)$ $\mathcal{N} = 4$ SYM living in the boundary of the AdS_5 space for large N[7]. This correspondence has been extended to several supergravity backgrounds and to different CFT. For a recent review, see [23]. This conjecture has attracted much attention and inspired thousands of scientific articles since its discovery. The main focus of this Ph.D. thesis is the study of both side of the conjecture. Specifically, we will focus on a superstring string theory with an $AdS_5 \times S^5$ target space using the pure spinor formalism and $\mathcal{N} = 2$ SCFTs, from a CFT point of view.

1.2 The String Side

As mentioned before, string theory is the more vivid candidate for quantum gravity. A drawback of bosonic string theory is the existence of a tachyon, a particle with negative $(\text{mass})^2$. When SUSY is introduced in the theory, this artefact is solved. There are two ways to add SUSY in string theory: in its worldsheet or in its target space, both leading to a theory with a ten dimensional target space. In the Ramond-Nevau-Schwarz (RNS) formulation of string theory, SUSY is added in the worldsheet. This theory can be quantized in an explicitly Lorentz covariant way, but consistency requires the use of the GSO projection, which avoids the scattering between space-time bosons to be mediated by worldsheet fermions. After the GSO projection is performed, a supersymmetryc spectrum also appears in the ten dimensional space. Unfortunately, it is not known how to quantize the RNS string in the presence of Ramond-Ramond fluxes, which are present in AdS backgrounds. In the Green-Schwarz (GS)

[7]It has been known since the seventies that $U(N)$ gauge theories with large N reproduce the amplitudes expected from string theory [22].

formalism, on the other hand, SUSY is introduced in the ten dimensional target space. Although this theory is explicitly covariant under space-time supersymmetry, it contains the same spectrum as the RNS superstring and it does not need a GSO projection in order to be consistent. Due to the mixing of first and second class constraints,[8] it is not known how to quantize GS superstring in an explicitly Lorentz covariant way, and quantization is achieved only in the light-cone gauge. This lack of covariant quantization in the GS superstring makes harder the computation of amplitudes and there are no explicit computations beyond one loop.

There is a third known prescription for superstring theory: the pure spinor string theory [26]. In this theory there is a constrained spinor, λ^α, which satisfies the constraint

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0, \tag{1.1}$$

where $\gamma^m_{\alpha\beta}$ are the ten-dimensional 16×16 Pauli matrices. (1.1) defines the pure spinor condition. The OPE for the constraint of the ten dimensional superstring in flat space [27],

$$d_\alpha = p_\alpha + \gamma_{\alpha\beta} \partial x_m \, \theta^\beta + \frac{1}{2} \gamma^m_{\alpha\beta} \, \gamma_{m\mu\nu} \theta^\beta \, \theta^\mu \, \partial\theta^\nu, \tag{1.2}$$

can be shown to be,

$$d_a(z) \, d_\beta(0) \sim \frac{2}{z} \gamma^m_{\alpha\beta} \Pi_m(z). \tag{1.3}$$

This OPE can be obtained because in the pure spinor formalism, x^m, θ^α, p_α are treated as a free system. We can introduce the nilpotent BRST charge,

$$Q = \oint dz \lambda^\alpha d_\alpha. \tag{1.4}$$

[8] A proposal for quantization of such systems was made in [24, 25].

The existence of the BRST charge[9] allows an explicitly Lorentz covariant quantization, unlike in the case of GS string, and unlike the RNS string there is no need of a mechanism similar to the GSO projection in order to obtain the tachyon free supersymmetric spectrum of the theory. A covariant quantization allows us to compute N-point tree amplitudes [29], for example. It is also possible to use the PS formalism to quantize other spaces where there are Ramond-Ramond fluxes and/or the light-cone quantization is not allowed. Although it is known that the massless spectrum of the superstring is the supergravity spectrum, a covariant description of this spectrum in the $AdS_5 \times S^5$ superstring is not known.

In order to see the complications arising in the $AdS_5 \times S^5$ superstring, let us study the bosonics string on a flat space-time target space. In the case of a flat space-time, the action for a scalar field is,

$$S_{\text{flat}} = \frac{1}{2} \int dV_{\text{flat}}\, \phi \,\Box\, \phi, \tag{1.5}$$

where dV_{flat} is the measure of the flat space volume. The corresponding EOM for this action is given by the Klein-Gordon equation for a massless particle,

$$\Box\, \phi = 0, \tag{1.6}$$

which admits a plane-wave solution,

$$\phi_{\text{flat}} \sim \exp\left(\mathrm{i}\, k \cdot x\right). \tag{1.7}$$

This wave solutions is exactly the form of the tachyonic field of the bosonic string, which is the lightest state of the bosonic string. Since p^2 is a Casimir operator in flat-space, it is easy to find the energy of any state by aplying p^2. Thus, it is easy to construct massless states by just applying $a_{\mu\nu}\, \bar{\partial} X^\mu\, \partial X^\nu$ to $\exp\left(\mathrm{i}\, k \cdot x\right)$, because each pair of X^μ terms added to

[9] The flat space-time constraint was written for simplicity. For generic supergravity background of Type IIB theories, the BRST charge can be constructed in a similar fashion [26, 28]

the tacyon field will raise the value of the energy of $\phi_{\mathbf{flat}}$ by two. The worldsheet conformal weight restriction of the operator will impose conditions to k^2, $k^\mu\, a_{\mu\,\nu}$ and to $a_{\mu\,\nu}$, and when applying p^2, we will be able to read off the mass of the state[10]. The operator $\exp\left(\mathrm{i}\, p\cdot x\right)$ can also be regarded as the responsible for the momentum transfer in the string interactions. In the $AdS_5\times S^5$ case, things are more complicated. Although the action for a scalar field is similar to S_{flat}, the solution to the EOM is more complicated than ϕ_{flat}. A solution for the scalar field on a $AdS_m\times S^m$ involves hypergeometric functions [30, 31]. A compact exact solution can be found for the conformally flat space $AdS_m\times S^m$ [30], which we will discuss in more detail in Chapter 3. Even if such simple expression exists, p^2 is no longer a Casimir operator in this curved space. This brings two more complications. First, the energy of the states can no longer be read from applying p^2 (in this case, the role of the energy is played by the eigenvalue of the target-space dilatation operator.) This, in turn, implies that we can no longer just apply an operator like $\bar\partial X^\mu\,\partial X^\nu$. Although we can, generally speaking, find the proper constraints in order to obtain the proper worldsheet conformal weight, the energy of such state cannot be easily computed. This is the main obstruction when trying to construct the physical. The second complication arises when trying to study interactions between strings, because there is no longer a notion for momentum transfer.

1.3 The CFT Side

The original AdS/CFT conjecture related a superstring theory with a $\mathcal{N}=4$ SYM theory. The integrability of the latter theory has lead to a great amount of results, ultimately leading to computation both at strong and weak coupling with great precision, for a review see [32]. However $\mathcal{N}=4$ SYM is not the only CFT dual to a superstring. There are also dualities for M-theories on $AdS_7\times S^4$ with a six dimensional $\mathcal{N}=(2,0)$ and on a $AdS_4\times S^7$ with a three dimensional ABJM theory [33]. For F-theory there are four dimensional dual theories with $\mathcal{N}=1,2$ SUSY [34].

[10] The correct answer is already know, and the $\bar\partial X^\mu\,\partial X^\nu$ operator was suggested from that knowledge.

As we mentioned before, the most symmetric theory in four dimensions is $\mathcal{N} = 4$ SYM. This is the only theory with 16 supercharges. The next most supersymmetric theories have $\mathcal{N} = 2$ sumerpsymmetry.[11] Several interesting phenomena arise in $\mathcal{N} = 2$ theories. For example, they present concrete examples of S-duality, confinement and monopole condensation [35, 36]. A remarkable structure of $SU(2)$ $\mathcal{N} = 2$ theories are the Seiberg-Witten curves [35, 36], which were latter generalized to other gauge groups in [37, 38]. Latter, a string origin for those objects was found [34] by proving F-theory with D3-branes. In recent years the number of $\mathcal{N} = 2$ SCFTs has grown considerably [39]. This ample amount of theories makes impractical the study of $\mathcal{N} = 2$ SCFTs one by one. One attempt to study these $\mathcal{N} = 2$ theories is to classify them [40, 41]. Another option is to obtain information that can be applied to all of them by using general arguments [42].

In the present thesis, we will develop methods that, we think, are the first step to construct general conformal blocks for $\mathcal{N} = 2$ SCFTs. This is a very important step before carrying out the bootstrap program for aforementioned theories.

1.4 Overview of this Thesis

In Chapters 2 and 3 we will focus on the string part of this thesis. We will first work out the worldsheet dilatation operator in Chapter 2, based on [43]. We will generalize the method developed in [44] to the coset action for the pure spinor string in a $AdS_5 \times S^5$ background. This will allow us to compute the anomalous dimension of several conserved currents by looking at the logarithmic divergences of composite operators. Those divergences can be related to the worldsheet dilatation operator. In Chapter 3 we will, following [45], construct a twistor action for the pure spinor string in the same spirit of [46]. This formalism will help us to construct a simple vertex operator related to a Noether current.

We will devot the remaining chapters 4 and 5 to the construction of the OPEs in $\mathcal{N} = 2$

[11] $\mathcal{N} = 3$ theories are not CPT invariant. When CPT is asked in $\mathcal{N} = 3$ theories, they are actually $\mathcal{N} = 4$ SYM.

SCFTs. First, we will focus on the stress tensor OPE in Chapter 4 in the same way as was treated in [47]. This OPE will help us to set a lower bound to the central charge for any interacting $\mathcal{N} = 2$ SCFT. We will then generalize the construction of the mixed OPEs between an $\mathcal{N} = 2$ stress-tensor multiplet, a chiral multiplet and a flavor current multiplet In Chapter 5. This will follow [48]

Each chapter will contain an introduction and conclusion in order to be completely self contained and therefore they can be read independently.

Chapter 2

Worldsheet dilatation operator for the AdS superstring

2.1 Introduction

During the last decade there was a great improvement in the understanding of $\mathcal{N} = 4$ super Yang-Mills theory due to integrability techniques, culminating in a proposal where the anomalous dimension of any operator can be computed at any coupling [49]. The crucial point of this advance was the realization that the computations of anomalous dimensions could be systematically done by studying the dilatation operator of the theory [50, 51]. For a general review and an extensive list of references, we recommend [32]. An alternative to the TBA approach not covered in [32], the Quantum Spectral Curve, was developed in [52, 53]. For some of its applications, including high loops computations, see [54, 55, 56, 57, 58, 59]

On the string theory side it is that known the world sheet sigma-model is classically integrable [60, 61]. However, it is not yet known how to fully quantize the theory, identifying all physical vertex operators and their correlation functions. In the case of the pure spinor string it is known that the model is conformally invariant at all orders of perturbation theory and that the non-local charges found in [61] exist in the quantum theory [62]. In a very interesting paper, [63] showed how to obtain the Y-system equations from the holonomy operator.

Another direction in which the pure spinor formalism was used with success was the quantization around classical configurations. In [64] it was shown that the semi-classical quantization of a large class of classical backgrounds agrees with the Green-Schwarz formalism. This was later generalized in [65, 66]. Previously, Mazzucato and one of the authors [67] attempted to use canonical quantization around a massive string solution to calculate the anomalous dimension of a member of the Konishi multiplet at strong coupling. Although the result agrees with both the prediction from integrability and Green-Schwarz formalism, this approach has several issues that make results unreliable [68].[1]

An alternative and more desirable approach is to use CFT techniques to study vertex operators and correlation functions since scattering amplitudes are more easily calculated using this approach. A first step is to identify physical vertex operators. Since the pure spinor formalism is based on BRST quantization, physical vertex operators should be in the cohomology of the BRST charge. For massless states, progress has been made in [69, 70, 71, 72]. For massive states the computation of the cohomology in a covariant way is a daunting task even in flat space [73].

A simpler requirement for physical vertices is that they should be primary operators of dimension zero for the unintegrated vertices and primaries of dimension $(1,1)$ in the integrated case. Massless unintegrated vertex operators in the pure spinor formalism are local operators with ghost number $(1,1)$ constructed in terms of zero classical conformal dimension fields [26]. So for them to remain primary when quantum corrections are taken into account, their anomalous dimension should vanish. Massless integrated vertices have zero ghost number and classical conformal dimension $(1,1)$. Therefore they will also be primaries when their anomalous dimension vanishes. Operators of higher mass level are constructed using fields with higher classical conformal dimension. For general mass level

[1] The authors would like to thank Martin Heinze for discussions on the subject.

n (where $n = 1$ corresponds to the massless states) the unintegrated vertex operators have classical conformal dimension $(n-1, n-1)$. If such vertex has anomalous dimension γ, the condition for it to be primary is $2n - 2 + \gamma = 0$. The case for integrated vertex operators is similar. For strings in flat space γ is always $\frac{\alpha' k^2}{2}$, which is the anomalous dimension of the plane wave $e^{ik\cdot X}$. This reproduces the usual mass level formula.

This task of computing γ can be made algorithmic in the same spirit as the four dimensional SYM case [50, 51]. However, here we are interested in finding the subset of operators satisfying the requirements described above. The value of the energy of the corresponding string state should come as the solution to an algebraic equation obtained from this requirement. However we do not expect the energy to be simply one of the parameters in the vertex operator. The proper way to identify the energy is to compute the conserved charge related to it and apply it to the vertex operator.

In this paper we intend to systematize the computation of anomalous dimensions in the worldsheet by computing all one-loop logarithmic short distance singularities in the product of operators with at most two derivatives. To find the answer for operators with more derivatives one simply has to compute the higher order expansion in the momentum of our basic propagator. We used the method applied by Wegner in [44] for the $O(n)$ model, but modified for the background field method. This was already used with success in [74, 75] for some $\mathbb{Z}_2$-super-coset sigma models. The pure spinor string is a $\mathbb{Z}_4$ coset and it has an interacting ghost system. This makes it more difficult to organize the dilatation operator in a concise expression and to find a solution to

$$\mathfrak{D} \cdot \mathcal{O} = 0. \tag{2.1}$$

We can select a set of "letters" $\{\phi^P\}$ among the basic fields of the sigma model, e.g. the AdS coordinates, ghosts and derivatives of these fields. Unlike the case of $\mathcal{N} = 4$ SYM, the worldsheet derivative is not one of the elements of the set, so fields with a different number of derivatives correspond to different letters. Then $\mathfrak{D}$ is of the form

$$\mathfrak{D} = \frac{1}{2}\mathfrak{D}^{PQ}\frac{\partial^2}{\partial\phi^P\partial\phi^Q}. \tag{2.2}$$

Local worldsheet operators are of the form

$$\mathcal{O} = V_{PQRST\cdots}\phi^P\phi^Q\phi^R\phi^S\phi^T\cdots, \tag{2.3}$$

the problem is to find $V_{ABCDE\cdots}$ such that $\mathcal{O}$ satisfy (2.1). Another important difference with the usual case is that the order of letters does not matter, so $\mathcal{O}$ is not a spin chain.

The problem of finding physical vertices satisfying this condition will be postponed to a future publication. Here we will compute $\mathfrak{D}$ and apply it to some local operators in the sigma model which should have vanishing anomalous dimension. The search for vertex operators in AdS using this approach was already discussed in [76] but without the contribution from the superspace variables. The author used the same "pairing" rules computed in [44].

This paper is organized as follows. In section 2 we describe the method used by Wegner in [44] for the simple case of the principal chiral field. This method consists of solving a Schwinger-Dyson equation in the background field expansion. In section 3 we explain how to apply these aforementioned method to the pure spinor AdS string case. The main derivation and results are presented in the Appendix B. Section 4 contains applications, where we use our results to compute the anomalous dimension of several conserved currents. Conclusions and further applications are in section 5.

2.2 Renormalization of operators in the principal chiral model

The purpose of this section is to review the computation of logarithmic divergences of operators in principal chiral models using the background field method. Although this is standard knowledge, the approach taken here is somewhat unorthodox so we include it for the sake of completeness. Also, the derivation of the full propagators in the case of $AdS_5 \times S^5$ is analogous to what is done in this section, so we omit their derivations.

Consider a principal model in some group G, with corresponding Lie algebra $\mathfrak{g}$, in two dimensions. The action is given by

$$S = -\frac{1}{2\pi\alpha^2}\int d^2z \,\mathrm{Tr}\,\partial g^{-1}\bar{\partial} g, \tag{2.4}$$

where α is the coupling constant and $g \in G$. Using the left-invariant currents $J = g^{-1}\partial g$ and defining $\sqrt{\lambda} = 1/\alpha^2$ we can also write

$$S = \frac{\sqrt{\lambda}}{2\pi}\int d^2z \,\mathrm{Tr}\, J\bar{J}. \tag{2.5}$$

The full one-loop propagator is derived from the Schwinger-Dyson equation

$$\langle \delta_z S\, \mathcal{O}(y)\rangle = \langle \delta_z \mathcal{O}(y)\rangle, \tag{2.6}$$

where δ_z is an arbitrary local variation of the fundamental fields and $\mathcal{O}(y)$ is a local operator. This equation comes from the functional integral definition of $\langle \cdots \rangle$. In order to be more explicit, let us consider a parametrization of g in terms of quantum fluctuations and a classical background $g = g_0 e^X$, where g_0 is the classical background, $X = X^a \mathfrak{J}_a$ and $\mathfrak{J}_a \in \mathfrak{g}$ are the generators of the algebra. Then a variation of g is given by $\delta g = g\delta X$, and $\delta X = \delta X^a \mathfrak{J}_a$ where we have the variation of the independent fields X^a. Also, the variation of some general operator $\mathcal{O}$ is $\delta\mathcal{O} = \frac{\delta\mathcal{O}}{\delta X^a}\delta X^a$. Then we can write the Schwinger-Dyson equation as

$$\left\langle \frac{\delta S}{\delta X^a(z)}\mathcal{O}(y)\right\rangle = \left\langle \frac{\delta \mathcal{O}(y)}{\delta X^a(z)}\right\rangle, \tag{2.7}$$

and now it is clear that this is a consequence of the identity

$$\int [DX] \frac{\delta}{\delta X^a(z)} \left(e^{-S}\mathcal{O}(y)\right) = 0. \tag{2.8}$$

In the case that $\mathcal{O}(y) = X(y)$ we get the Schwinger-Dyson equation for the propagator

$$\left\langle \frac{\delta S}{\delta X^a(z)} X^b(y) \right\rangle = \delta_a^b \delta^2(y-z). \tag{2.9}$$

This is a textbook way to get the equation for the propagator in free field theories and our goal here is to solve this equation for the interacting case at one loop order. The perturbative expansion of the action is done using the background field method. A fixed background g_0 is chosen and the quantum fluctuation is defined as $g = g_0 e^X$. The expansion of the current is given by

$$J = e^{-X} \boldsymbol{J} e^X + e^{-X} \partial e^X \tag{2.10}$$

where $\boldsymbol{J} = g_0^{-1}\partial g_0$ is the background current. At one loop order only quadratic terms in the quantum field expansion contribute and, as usual, linear terms cancel by the use of the background equation of motion. This means that we can separate the relevant terms action in two pieces $S = S_{(0)} + S_{(2)}$. Furthermore, $S_{(2)}$ contains the kinetic term plus interactions with the background. So we have

$$S = S_{(0)} + S_{kin} + S_{int}. \tag{2.11}$$

If we insert this into (2.9) the terms that depend purely on the background cancel and we are left with

$$\left\langle \frac{\delta S_{kin}}{\delta X^a(z)} X^b(y) + \frac{\delta S_{int}}{\delta X^a(z)} X^b(y) \right\rangle = \delta_a^b \delta^2(y-z). \tag{2.12}$$

Since $\frac{\delta S_{kin}}{\delta X^a(z)} = -\frac{\sqrt{\lambda}}{2\pi}\partial\bar{\partial} X^a(z)$ and $\frac{\delta S_{int}}{\delta X^a(z)}$ is linear in quantum fields we can write

$$-\frac{\sqrt{\lambda}}{\pi}\partial_z\bar{\partial}_{\bar{z}}\langle X^a(z)X^b(y)\rangle + \int d^2w \frac{\delta^2 S_{int}}{\delta X^c(w)\delta X^d(z)}\langle X^c(w)X^b(y)\rangle\eta^{ad} = \eta^{ab}\delta^2(y-z). \quad (2.13)$$

Finally, this is the equation that we have to solve. It is an integral equation for $\langle X^a(z)X^b(y)\rangle = G^{ab}(z,y)$ which is the one-loop corrected propagator. The interacting part of the action is

$$S_{int} = \frac{\sqrt{\lambda}}{2\pi}\int d^2z \left[-\frac{1}{2}\mathrm{Tr}([\partial X, X]\bar{\boldsymbol{J}}) - \frac{1}{2}\mathrm{Tr}([\bar{\partial}X, X]\boldsymbol{J})\right], \quad (2.14)$$

where the boldface fields stand for the background fields.

Now we calculate[2]

$$\frac{\delta^2 S_{int}}{\delta X^c(w)\delta X^a(z)} = \frac{\sqrt{\lambda}}{2\pi}[\partial_w\delta^2(w-z)\mathrm{Tr}([T_c,T_a]\bar{\boldsymbol{J}}) + \bar{\partial}_w\delta^2(w-z)\mathrm{Tr}([T_c,T_a]\boldsymbol{J})], \quad (2.15)$$

which is symmetric under exchange of (a,z) and (c,w), as expected. We define $f_c^{ab} = f_{cd}^b\eta^{da}$. So we get the following equation for the propagator

$$\partial_z\bar{\partial}_z G^{ab}(z,y) = -\frac{\pi}{\sqrt{\lambda}}\eta^{ab}\delta^2(y-z) + \frac{f_{ce}^a}{2}\left(\partial_z G^{cb}(z,y))\bar{\boldsymbol{J}}^e + \bar{\partial}_z G^{cb}(z,y)\boldsymbol{J}^e\right). \quad (2.16)$$

Performing the Fourier transform

$$G^{ac}(z,k) = \int d^2y e^{-ik\cdot(z-y)}G^{ac}(z,y), \quad (2.17)$$

we finally get

[2] Using the equation of motion for the background $\partial\bar{\boldsymbol{J}} + \bar{\partial}\boldsymbol{J} = 0$.

$$\begin{aligned} G^{ab}(z,k) =& \frac{\eta^{ab}}{\sqrt{\lambda}} \frac{\pi}{|k|^2} + \frac{\Box}{|k|^2} G^{ab}(z,k) + i\frac{\partial}{k} G^{ab}(z,k) + i\frac{\bar{\partial}}{\bar{k}} G^{ab}(z,k) \\ & - f^a_{ce} \bar{\boldsymbol{J}}^e \left(\frac{i}{2\bar{k}} + \frac{\partial_z}{2|k|^2} \right) G^{cb}(z,k) - f^a_{ce} \boldsymbol{J}^e \left(\frac{i}{k} + \frac{\bar{\partial}_z}{2|k|^2} \right) G^{cb}(z,k). \end{aligned} \tag{2.18}$$

The dependence on one of the coordinates remains because the presence of background fields breaks translation invariance on the worldsheet. We can solve the equation above iteratively in inverse powers of k. The first few contributions are given by

$$\begin{aligned} G^{ab}(z,k) =& \frac{\eta^{ab}}{\sqrt{\lambda}} \frac{\pi}{|k|^2} - \frac{i\pi f^a_{ce}}{2\sqrt{\lambda}|k|^2} \eta^{cb} \left(\frac{\bar{\boldsymbol{J}}^e}{\bar{k}} + \frac{\boldsymbol{J}^e}{k} \right) \\ & - \frac{\pi}{4\sqrt{\lambda}} f^a_{ce} f^c_{df} \eta^{db} \frac{1}{|k|^2} \left(\frac{1}{|k|^2} (\boldsymbol{J}^e \bar{\boldsymbol{J}}^f + \bar{\boldsymbol{J}}^e \boldsymbol{J}^f) + \frac{1}{\bar{k}^2} \bar{\boldsymbol{J}}^e \bar{\boldsymbol{J}}^f + \frac{1}{k^2} \boldsymbol{J}^e \boldsymbol{J}^f \right) \\ & + \frac{\pi}{2\sqrt{\lambda}} \frac{\eta^{db} f^a_{df}}{|k|^2} \left(\frac{1}{\bar{k}^2} \bar{\partial} \bar{\boldsymbol{J}}^f + \frac{1}{k^2} \partial \boldsymbol{J}^f \right) + \dots \end{aligned} \tag{2.19}$$

With this solution we can finally do the inverse Fourier transform,

$$G^{ac}(z,y) = \int \frac{d^2k}{4\pi^2} e^{ik\cdot(z-y)} G^{ac}(z,k), \tag{2.20}$$

to calculate $G^{ac}(z,y)$. If we are only interested in the divergent part of the propagator we can already set $z = y$. Furthermore, selecting only the divergent terms in the momentum integrals we get

$$\langle X^a(z) X^c(z) \rangle = \frac{\mathrm{I}\pi}{\sqrt{\lambda}} \eta^{ac}, \tag{2.21}$$

$$\langle X^a(z) \partial X^c(z) \rangle = -\frac{\mathrm{I}\pi}{2\sqrt{\lambda}} \eta^{dc} f^a_{de} \boldsymbol{J}^e, \tag{2.22}$$

$$\langle X^a(z) \bar{\partial} X^c(z) \rangle = -\frac{\mathrm{I}\pi}{2\sqrt{\lambda}} \eta^{dc} f^a_{de} \bar{\boldsymbol{J}}^e, \tag{2.23}$$

$$\langle X^a(z) \partial\bar{\partial} X^b(z) \rangle = \frac{\mathrm{I}\pi}{4\sqrt{\lambda}} \eta^{db} f^c_{df} f^a_{ce} \left(\boldsymbol{J}^e \bar{\boldsymbol{J}}^f + \bar{\boldsymbol{J}}^e \boldsymbol{J}^f \right), \tag{2.24}$$

$$\langle X^a(z)\partial\partial X^b(z)\rangle = -\frac{\mathrm{I}\pi}{2\sqrt{\lambda}}\eta^{cb}f^a_{ce}\partial \boldsymbol{J}^e + \frac{\mathrm{I}\pi}{4\sqrt{\lambda}}\eta^{db}f^a_{ce}f^c_{df}\boldsymbol{J}^e\boldsymbol{J}^f, \tag{2.25}$$

$$\langle X^a(z)\bar\partial\bar\partial X^c(z)\rangle = -\frac{\mathrm{I}\pi}{2\sqrt{\lambda}}\eta^{cb}f^a_{ce}\bar\partial \bar{\boldsymbol{J}}^e + \frac{\mathrm{I}\pi}{4\sqrt{\lambda}}\eta^{db}f^a_{ce}f^c_{df}\bar{\boldsymbol{J}}^e\bar{\boldsymbol{J}}^f, \tag{2.26}$$

where

$$\mathrm{I} = -\frac{1}{2\pi\epsilon} = \lim_{x\to y}\int \frac{d^{2+\epsilon}}{4\pi^2}\frac{\epsilon^{ik(x-y)}}{|k|^2} \tag{2.27}$$

in $d = 2+\epsilon$ dimensions, using the standard dimensional regularization [44]. Since $\partial\langle X^a\partial X^c\rangle = \langle\partial X^a\partial X^c\rangle + \langle X^a\partial^2 X^c\rangle$ we can further compute

$$\langle\partial X^a(z)\partial X^b(z)\rangle = -\frac{\mathrm{I}\pi}{4\sqrt{\lambda}}(f^a_{ce}f^c_{df}\eta^{db}\boldsymbol{J}^e\boldsymbol{J}^f), \tag{2.28}$$

$$\langle\bar\partial X^a(z)\partial X^b(z)\rangle = -\frac{\mathrm{I}\pi}{2\sqrt{\lambda}}\eta^{cb}f^a_{ce}\bar\partial \boldsymbol{J}^e - \frac{\mathrm{I}\pi}{4\sqrt{\lambda}}\eta^{db}f^c_{df}f^a_{ce}(\bar{\boldsymbol{J}}^e\boldsymbol{J}^f + \bar{\boldsymbol{J}}^f\boldsymbol{J}^e). \tag{2.29}$$

From now on $\langle\cdot\rangle$ will mean only the logarithmically divergent part of the expectation value. A simple way to extract this information is by defining

$$\langle\mathcal{O}\rangle = \frac{1}{2}\int d^2z d^2y \frac{\delta^2\mathcal{O}}{\delta X^a(z)\delta X^b(y)}\langle X^a(z)X^b(y)\rangle, \tag{2.30}$$

for any local operator $\mathcal{O}$. Furthermore, we define

$$\langle\mathcal{O}\,,\mathcal{O}'\rangle = \int d^2z d^2y \frac{\delta\mathcal{O}}{\delta X^a(z)}\frac{\delta\mathcal{O}'}{\delta X^b(y)}\langle X^a(z)X^b(y)\rangle. \tag{2.31}$$

Following [76] we will call it "pairing" rules. For local operators these two definitions always give two delta functions, effectively setting all fields at the same point. So the computation of $\langle\cdot\rangle$ can be summarized as

$$\langle\mathcal{O}\rangle = \frac{1}{2}\langle X^a X^b\rangle\frac{\partial^2}{\partial X^a\partial X^b}\mathcal{O} = \mathfrak{D}\mathcal{O}, \tag{2.32}$$

where

$$\mathfrak{D} = \frac{1}{2}\langle X^a X^b\rangle \frac{\partial^2}{\partial X^a \partial X^b} \tag{2.33}$$

is the dilatation operator. We can also define $\langle \cdot\,,\cdot\rangle$ as

$$\langle \mathcal{O}\,,\mathcal{O}'\rangle = \langle X^a X^b\rangle \frac{\partial \mathcal{O}}{\partial X^a}\frac{\partial \mathcal{O}'}{\partial X^b}. \tag{2.34}$$

With the above definitions, the divergent part of any product of local operators at the same point can be computed using.

$$\langle \mathcal{O}\mathcal{O}'\rangle = \langle \mathcal{O}\rangle \mathcal{O}' + \mathcal{O}\langle \mathcal{O}'\rangle + \langle \mathcal{O}\,,\mathcal{O}'\rangle. \tag{2.35}$$

Several known results can be derived using this simple set of rules. Following this procedure in the case of the symmetric space $SO(N+1)/SO(N)$ gives the same results obtained by Wegner [44] using a different method.

2.3 Dilatation operator for the $AdS_5 \times S^5$ superstring

In this section we will apply the same technique to the case of the pure spinor AdS string. We begin with a review of the pure spinor description, pointing out the differences between this model and the principal chiral model, and then describe the main steps of the computation.

2.3.1 Pure spinor AdS string

The pure spinor string [73, 61, 62] in AdS has the same starting point as the Metsaev-Tseyltin [77]. The maximally supersymmetric type IIB background $AdS_5 \times S^5$ is described by the supercoset

$$\frac{G}{H} = \frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}. \tag{2.36}$$

The pure spinor action is given by

$$S_{PS} =\frac{R^2}{2\pi}\int d^2z \mathrm{STr}\left[\frac{1}{2}J_2\bar{J}_2+\frac{1}{4}J_1\bar{J}_3+\frac{3}{4}\bar{J}_1J_3+\omega\bar{\nabla}\lambda+\hat{\omega}\nabla\hat{\lambda}-N\hat{N}\right], \tag{2.37}$$

where

$$\nabla\cdot =\partial\cdot+[J_0,\cdot], \qquad N =\{\omega,\lambda\}, \qquad \hat{N} =\{\hat{\omega},\hat{\lambda}\}. \tag{2.38}$$

There are several difference between the principal chiral model action and (2.37). First, the model is coupled to ghosts. The pure spinor action also contains a Wess-Zumino term, and the global invariant current J belongs to the $\mathfrak{psu}(2,2|4)$ algebra, which is a graded algebra, with grading 4. Thus we split the current as $J = A + J_1 + J_2 + J_3$, where $A = J_0$ belongs to the algebra of the quotient group $H = SO(1,4) \times SO(5)$. The notation that we use for currents of different grade is

$$J_0 =J_0^iT_i \quad ; \quad J_1 = J_1^\alpha T_\alpha \quad ; \quad J_2 = J_2^m T_m \quad ; \quad J_3 = J_3^{\hat{\alpha}}T_{\hat{\alpha}}. \tag{2.39}$$

The ghosts fields are defined as

$$\lambda =\lambda^A T_A \quad ; \quad \omega = -\omega_A\eta^{A\hat{A}}T_{\hat{A}} \quad ; \quad \hat{\lambda} = \hat{\lambda}^{\hat{A}}T_{\hat{A}} \quad ; \quad \hat{\omega} = \hat{\omega}_{\hat{B}}\eta^{B\hat{B}}T_B. \tag{2.40}$$

Note that A and A' indices on the ghosts mean α and $\hat{\alpha}$, but we will use a different letter in order to make it easier to distinguish which terms come from ghosts and which come from the algebra. The pure spinor condition can be written as

$$\{\lambda,\lambda\} = \{\hat{\lambda},\hat{\lambda}\} = 0. \tag{2.41}$$

Following the principal chiral model example, we expand g around a classical background g_0 using the $g = g_0e^X$ parametrization. It is worth noting that $X = x_0 + x_1 + x_2 + x_3$

belongs to the $\mathfrak{psu}(2,2|4)$ algebra, but we can use the coset property to fix $x_0 = 0$. With this information the quantum expansion of the left invariant current is

$$\begin{aligned}
A =& \boldsymbol{A} + \sum_{i=1}^{3}\left([\boldsymbol{J}_i, x_{4-i}] + \frac{1}{2}[\nabla x_i, x_{4-i}]\right) + \sum_{i,j=1}^{3}\left[[\boldsymbol{J}_i, x_j], x_{8-i-j}\right],\\
J_l =& \boldsymbol{J}_l + \nabla x_l + \sum_{i=1}^{3}\left([\boldsymbol{J}_i, x_{4+l-i}] + \frac{1}{2}[\nabla x_i, x_{4+l-i}]\right) + \sum_{i,j=1}^{3}\left[[\boldsymbol{J}_i, x_j], x_{8+l-i-j}\right],\\
\lambda =& \boldsymbol{\lambda} + \delta\lambda,\\
\omega =& \boldsymbol{\omega} + \delta\omega,\\
\hat{\lambda} =& \hat{\boldsymbol{\lambda}} + \delta\hat{\lambda},\\
\hat{\omega} =& \hat{\boldsymbol{\omega}} + \delta\hat{\omega}.
\end{aligned} \tag{2.42}$$

Where we take $x_0 = 0$ as mentioned before, and we used $g_0^{-1}\partial g_0 = \boldsymbol{J} = \boldsymbol{A} + \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3$. The boldface terms stand for the background term, both for the currents and for the ghost fields.

Using all this information inside the action we get

$$S_{PS} = \frac{R^2}{2\pi}\int d^2z\left[\frac{1}{2}\nabla x_2^m \bar{\nabla} x_2^n \eta_{mn} - \nabla x_1^{\alpha}\bar{\nabla} x_3^{\hat{\alpha}}\eta_{\alpha\hat{\alpha}} + \delta\omega_A \bar{\partial}\delta\lambda^A + \delta\hat{\omega}_{\hat{A}}\partial\delta\hat{\lambda}^{\hat{A}}\right] + S_{int}. \tag{2.43}$$

The full expansion can be found in the Appendix C. In order to compute the logarithmic divergences, we need to generalize the method explained in section 2 for a coset model with ghosts. The following subsection is devoted to explain this generalization.

2.3.2 General coset model coupled to ghosts

In this subsection we generalize the method of Section 2 to the case of a general coset G/H and then specialize for the pure spinor string case. We will denote the corresponding algebras $\mathfrak{g}$ and $\mathfrak{h}$, where $\mathfrak{h}$ should be a subalgebra of $\mathfrak{g}$. The generators of $\mathfrak{g} - \mathfrak{h}$ will be denoted by T_a where $a = 1$ to $\dim_{\mathfrak{g}} - \dim_{\mathfrak{h}}$ and the generators of $\mathfrak{h}$ will be denoted by T_i where $i = 1$

to $\dim_{\mathfrak{h}}$. We also include a pair of first order systems (λ^A, ω_B) and $(\hat{\lambda}^{A'}, \hat{\omega}_{B'})$ transforming in two representations $(\Gamma^{iB}_{A}, \Gamma^{iB'}_{A'})$ of $\mathfrak{h}$. We will assume that the algebra $\mathfrak{g}$ has the following commutation relations

$$[T_a, T_b] = f^c_{ab} T_c + f^i_{ab} T_i, \quad [T_a, T_i] = f^b_{ai} T_b, \quad [T_i, T_j] = f^k_{ij} T_k, \tag{2.44}$$

where $f^c_{ab} \neq 0$ for a general coset and $f^c_{ab} = 0$ if there is a $\mathbb{Z}_2$ symmetry, i.e., G/H is a symmetric space. As in the usual sigma model $g \in G/H$ and the currents $J = g^{-1}\partial g$ are invariant by left global transformations in G. We can decompose $J = J^a T_a + A^i T_i$ where $J^a T_a \in \mathfrak{g} - \mathfrak{h}$ and $A^i T_i \in \mathfrak{h}$. With this decomposition K transforms in the adjoint representation of $\mathfrak{h}$ and A transforms as a connection. We will also allow a quartic interaction in the first order sector. Defining $N^i = \lambda^A \Gamma^{iB}_{A} \omega_B$ and $\hat{N}^i = \hat{\lambda}^{A'} \Gamma^{iB'}_{A'} \hat{\omega}_{B'}$, the interaction will be $\beta N^i \hat{N}_i$ where β is a new coupling constant that in principle is not related with the sigma model coupling.

The total action is given by

$$S = \int d^2 z \left(\mathrm{Tr}\ (J - A)(\bar{J} - \bar{A}) + \omega_A \bar{\nabla} \lambda^A + \hat{\omega}_{A'} \nabla \lambda^{A'} + \beta N^i \hat{N}_i \right), \tag{2.45}$$

where $(\nabla, \bar{\nabla}) = (\partial - A^i \Gamma_{i B'}^{A'}, \bar{\partial} - \bar{A}^i \Gamma_{i B}^{A})$ are the covariant derivatives for the first order system ensuring gauge invariance.

The background field expansion is different if we are in a general coset or a symmetric space. Since we want to generalize the results to the case of $AdS_5 \times S^5$, we will use a notation that keeps both types of interactions. Again, the quantum coset element is written as $g = g_0 e^X$ where g_0 is the classical background and $X = X^a T_a$ are the quantum fluctuations.

Up to quadratic terms in the quantum fluctuation the expansion of the action is

$$
\begin{aligned}
S = & S_0 + \int d^2z \Big(\eta_{ab}\nabla X^a \bar{\nabla} X^b - Z_{abc}\boldsymbol{J}^a X^b \bar{\nabla} X^c - \bar{Z}_{abc}\bar{\boldsymbol{J}}^a X^b \nabla X^c + R_{abcd}\boldsymbol{J}^a \bar{\boldsymbol{J}}^b X^c X^d \\
& +\delta\omega_A \bar{\partial}\delta\lambda^A + \delta\hat{\omega}_{A'}\partial\delta\hat{\lambda}^{A'} + \bar{\boldsymbol{A}}^i N_i + \boldsymbol{A}^i \hat{N}_i + \beta\left(\{\delta\lambda, \boldsymbol{\omega}\} + \{\boldsymbol{\lambda}, \delta\omega\}\right)^i \left(\{\delta\hat{\lambda}, \hat{\boldsymbol{\omega}}\} + \{\hat{\boldsymbol{\lambda}}, \delta\hat{\omega}\}\right)_i \Big),
\end{aligned}
\tag{2.46}
$$

where the covariant derivatives on X are $(\partial - [A, \cdot\,], \bar{\partial} - [\bar{A}, \cdot\,])$. The tensors $(Z_{abc}, \bar{Z}_{abc}, R_{abcd})$ appearing above are model dependent. In the case of a symmetric space $Z = \bar{Z} = 0$ and $R_{abcd} = f^i_{ab} f_{icd}$. In the general coset case $Z_{abc} = \bar{Z}_{abc} = \frac{1}{2} f_{abc}$. If there is a Wess-Zumino term, the values of Z_{abc} and $\bar{Z}_{abc}$ can differ. Since we want to do the general case, we will not substitute the values of these tensor until the end of the computations. In the action above the quantum connections have the following expansion

$$
A^i = \boldsymbol{A}^i + f^i_{ab}\boldsymbol{J}^a X^b + \frac{1}{2} f^i_{ab} \nabla X^a X^b + W^i_{abc}\boldsymbol{J}^a X^b X^c + \cdots \tag{2.47}
$$

$$
\bar{A}^i = \bar{\boldsymbol{A}}^i + f^i_{ab}\bar{\boldsymbol{J}}^a X^b + \frac{1}{2} f^i_{ab} \bar{\nabla} X^a X^b + W^i_{abc}\bar{\boldsymbol{J}}^a X^b X^c + \cdots \tag{2.48}
$$

where $W^i_{abc} = \frac{1}{2} f^d_{ab} f^i_{dc}$ for a general coset and vanishes for a symmetric space.

To proceed, we have to compute the second order variation of the action with respect to the quantum fields. The difference this time is that there are many more couplings, so we expect a system of coupled Schwinger-Dyson equations, corresponding to each possible corrected propagator. For example, in the free theory approximation there is no propagator between the sigma model fluctuation and the first order system, but due to the interactions there we may have corrected propagators between them.

Since a propagator is not a gauge invariant quantity, it can depend on gauge dependent combinations of the background gauge fields $(A^i, \bar{A}^i)$. Furthermore, since we have chiral fields transforming in two different representations of $\mathfrak{h}$ it is possible that the quantum theory has anomalies. In the case of the $AdS_5 \times S^5$ string sigma-model it was argued by Berkovits

[62] that there is no anomaly for all loops. An explicit one loop computation was done in [78]. Therefore it is safe to assume that the background gauge fields only appear in physical quantities in a gauge invariant combination. The simplest combination of this type is $\mathrm{Tr}[\nabla, \bar{\nabla}]^2$. Since the classical conformal dimension of this combination is four and so far we are interested in operators of classical conformal dimension 0 and 2, we can safely ignore all interactions with $(A^i, \bar{A}^i)$.

We will assume a linear quantum variation of the first order system, e.g., $\lambda^A \to \boldsymbol{\lambda}^A + \delta\lambda^A$. Instead of introducing more notation and a cumbersome interaction Lagrangian, we will simply compute the variations of these fields in the action and set to their background values the remaining fields.

With all these simplifications and constraints in mind, let us start constructing the Schwinger-Dyson equations. First we compute all possible non-vanishing second variations of the action

$$\begin{aligned}\frac{\delta^2 S_{int}}{\delta X^a \delta X^b} &= \delta^2(z-w)\left[\boldsymbol{J}^c\bar{\boldsymbol{J}}^d(R_{cdab}+R_{cdba}) + \boldsymbol{N}_i\bar{\boldsymbol{J}}^c(W^i_{cab}+W^i_{cba}) + \hat{\boldsymbol{N}}_i\boldsymbol{J}^c(W^i_{cab}+W^i_{cba})\right]\\ &\quad - \partial_w\delta^2(z-w)[\bar{Z}_{cab}\bar{\boldsymbol{J}}^c + f^i_{ab}\hat{\boldsymbol{N}}_i] - \bar{\partial}_w\delta^2(z-w)[Z_{cab}\boldsymbol{J}^c + f^i_{ab}\boldsymbol{N}_i], \qquad (2.49a)\\ \frac{\delta^2 S_{int}}{\delta\lambda^A\delta\omega_B} &= \delta^2(z-w)\beta\Gamma^{iB}_{\ A}\hat{\boldsymbol{N}}_i, \qquad (2.49b)\\ \frac{\delta^2 S_{int}}{\delta\lambda^A\delta X^a} &= \delta^2(z-w)(\Gamma_i\boldsymbol{\omega})_A f^i_{ba}\bar{\boldsymbol{J}}^b, \qquad (2.49c)\\ \frac{\delta^2 S_{int}}{\delta\lambda^A\delta\hat{\lambda}^{B'}} &= \delta^2(z-w)\beta(\Gamma^i\boldsymbol{\omega})_A(\hat{\Gamma}_i\hat{\boldsymbol{\omega}})_{B'}, \qquad (2.49d)\\ \frac{\delta^2 S_{int}}{\delta\lambda^A\delta\hat{\omega}_{B'}} &= \delta^2(z-w)\beta(\Gamma^i\boldsymbol{\omega})_A(\hat{\boldsymbol{\lambda}}\hat{\Gamma}_i)^{B'}, \qquad (2.49e)\\ \frac{\delta^2 S_{int}}{\delta\hat{\lambda}^{A'}\delta\hat{\omega}_{B'}} &= \delta^2(z-w)\beta\boldsymbol{N}^i\hat{\Gamma}_{iA'}^{\ B'}, \qquad (2.49f)\\ \frac{\delta^2 S_{int}}{\delta\hat{\lambda}^{A'}\delta X^a} &= \delta^2(z-w)(\hat{\Gamma}_i\hat{\boldsymbol{\omega}})_{A'}f^i_{ba}\boldsymbol{J}^b, \qquad (2.49g)\\ \frac{\delta^2 S_{int}}{\delta\hat{\lambda}^{A'}\delta\omega_B} &= \delta^2(z-w)\beta(\boldsymbol{\lambda}\Gamma^i)^B(\hat{\Gamma}_i\hat{\boldsymbol{\omega}})_{A'}, \qquad (2.49h)\\ \frac{\delta^2 S_{int}}{\delta X^a\delta\omega_B} &= \delta^2(z-w)(\boldsymbol{\lambda}\Gamma_i)^B f^i_{ba}\bar{\boldsymbol{J}}^b, \qquad (2.49i)\end{aligned}$$

$$\frac{\delta^2 S_{int}}{\delta X^a \delta\hat{\omega}_{B'}} = \delta^2(z-w)(\hat{\boldsymbol{\lambda}}\hat{\Gamma}_i)^{B'} f^i_{ba}\boldsymbol{J}^b, \tag{2.49j}$$

$$\frac{\delta^2 S_{int}}{\delta\omega_A \delta\hat{\omega}_{B'}} = \delta^2(z-w)\beta(\boldsymbol{\lambda}\Gamma^i)^A(\hat{\boldsymbol{\lambda}}\hat{\Gamma}_i)^{B'}. \tag{2.49k}$$

We are going to denote these second order derivatives generically as $I_{\Sigma\Lambda}(z,w)$ where Σ and Λ can be any of the indices $({}^a, {}^A, {}_B, {}^{A'}, {}_{B'})$. Also, the quantum fields will be denoted by Φ^Σ(z). With this notation the Schwinger-Dyson equations are

$$\langle\frac{\delta S_{kin}}{\delta\Phi^\Lambda(z)}\Phi^\Sigma(y)\rangle + \int d^2w \frac{\delta^2 S_{int}}{\delta\Phi^\Upsilon(w)\delta\Phi^\Lambda(z)}\langle\Phi^\Upsilon(w)\Phi^\Sigma(y)\rangle = \delta^\Sigma_\Lambda\delta(z-y). \tag{2.50}$$

Note that the only non-vanishing components of δ^Σ_Λ are η^{ab}, δ^A_B and $\delta^{A'}_{B'}$. Since the type and the position of the indices completely identity the field, the propagators are going to be denoted by $G^{\Sigma\Lambda}(z,y) = \langle\Phi^\Sigma(z)\Phi^\Lambda(y)\rangle$. Since we five different types of fields, we have fifteen coupled Schwinger-Dyson equations to solve. Again we have to make a simplification. Interpreting $(\lambda^A, \hat{\lambda}^{A'})$ as left and right moving ghosts and knowing that in the pure spinor superstring unintegrated vertex operators have ghost number $(1,1)$ with respect to $(G, \hat{G})$, we will concentrate on only four corrected propagators $\langle X^a(z)X^b(y)\rangle$, $\langle X^a(z)\lambda^A(y)\rangle$, $\langle X^a(z)\hat{\lambda}^{A'}(y)\rangle$ and $\langle\lambda^A(z)\hat{\lambda}^{A'}(y)\rangle$. As in the principal chiral model case we are going to solve the Schwinger-Dyson equations first in momentum space. It is useful to note that since we will solve this equations in inverse powers of k, the first contributions to the corrected propagators will have the form

$$\langle X^c X^a\rangle \approx \frac{\eta^{ca}}{|k|^2}, \qquad \langle\omega_A\lambda^B\rangle \approx \frac{\delta^B_A}{\bar{k}}, \qquad \langle\hat{\omega}_{A'}\hat{\lambda}^{B'}\rangle \approx \frac{\delta^{B'}_{A'}}{k}. \tag{2.51}$$

Regarding (A, A') as one type of index we can arrange the whole Schwinger-Dyson equation into a matrix notation with three main blocks. Doing the same Fourier transform as before we get a matrix equation that can be solved iteratively

$$G^\Upsilon_\Sigma = I^\Upsilon_\Sigma + (F_{\Sigma\Gamma} + \Delta_{\Sigma\Gamma})G^{\Gamma\Upsilon}, \tag{2.52}$$

where

$$I_{\Sigma}^{\Upsilon} = \begin{pmatrix} \frac{\delta_b^a}{|k|^2} & 0 & 0 \\ 0 & 0 & -i\frac{\delta_B^A}{k} \\ 0 & i\frac{\delta_A^B}{k} & 0 \end{pmatrix}, \tag{2.53}$$

$$F_{\Sigma\Gamma} = \frac{\delta^2 S_{int}}{\delta\Phi^{\Sigma}\delta\Phi^{\Gamma}}, \qquad \Delta_{\Sigma\Gamma} = \begin{pmatrix} \frac{\partial\bar{\partial}}{|k|^2} + i\frac{\partial}{k} + i\frac{\bar{\partial}}{\bar{k}} & 0 & 0 \\ 0 & -i\frac{\partial}{k} & 0 \\ 0 & 0 & i\frac{\partial}{k} \end{pmatrix}. \tag{2.54}$$

All elements of the interaction matrix $F_{\Sigma\Gamma}$ are shown in Appendix C. As in section 2, the solution to equation (2.52) is computed iteratively

$$G^{(0)}{}_{\Sigma}^{\Upsilon} = I_{\Sigma}^{\Upsilon}, \qquad G^{(1)}{}_{\Sigma}^{\Upsilon} = F_{\Sigma}^{\Gamma} I_{\Gamma}^{\Upsilon}, \tag{2.55}$$

and so on for higher inverse powers of k.

2.3.3 Pairing rules

As discussed in the introduction and Section 2, the computation of the divergent part of any local operator can be summarized by the pairing rules of a set of letters $\{\phi^P\}$. The complete set of these pairing rules can be found in the Appendix C. If we choose a set of letters such that $\langle\phi^P\rangle = 0$, then the divergent part of the product of two letters is simply

$$\langle\phi^P\phi^Q\rangle = \langle\phi^P,\ \phi^Q\rangle. \tag{2.56}$$

We computed the momentum space Green function up to quartic inverse power of momentum so we must restrict our set of letters to fundamental fields up to classical dimension one. The convenient set of letter we will use is

$$\{\phi^P\} = \{x_2^a, x_1^\alpha, x_3^{\hat\alpha}, J_2^a, J_1^\alpha, J_3^{\hat\alpha}, J_0^i, \bar{J}_2^a, \bar{J}_1^\alpha, \bar{J}_3^{\hat\alpha}, \bar{J}_0^i, \lambda^A, \omega_A, \hat\lambda^{\hat A}, \hat\omega_{\hat A}, N^i, \hat{N}^i\}. \tag{2.57}$$

If we extend the computation to take into account operators with more than two derivatives the set of letters has to be extended to include them. The matrix elements of the dilation operator $\mathfrak{D}^{PQ} = \langle \phi^P, \phi^Q \rangle$ are the full set of pairings described in Appendix C.4. To avoid cumbersome notation, the pairing rules are written contracting with the corresponding $\mathfrak{psu}(2,2|4)$ generator. The computations done in next section are a straightforward application of the differential operator

$$\mathfrak{D} = \frac{1}{2}\mathfrak{D}^{PQ}\frac{\partial^2}{\partial\phi^P\partial\phi^Q} \tag{2.58}$$

on a a local operator of the form $\mathcal{O} = V_{PQRST\cdots}\phi^P\phi^Q\phi^R\phi^S\phi^T\cdots$.

2.4 Applications

In this section we use our results to prove that certain important operators in the pure spinor sigma model are not renormalized. The operators we choose are stress energy tensor, the conserved currents related to the global $PSU(2,2|4)$ symmetry and the composite b-ghost. All these operators are a fundamental part of the formalism and it is a consistency check that they are indeed not renormalized. All the computations bellow are an application of the differential operator (2.58). We use the notation $\langle\mathcal{O}\rangle = \mathfrak{D}\cdot\mathcal{O}$.

2.4.1 Stress-energy tensor

The holomorphic and anti-holomorphic stress-energy tensor for (2.37) are given by

$$T = \mathrm{STr}\left(\frac{1}{2}J_2J_2 + J_1J_3 + \omega\nabla\lambda\right), \tag{2.59}$$

$$\bar{T} = \mathrm{STr}\left(\frac{1}{2}\bar{J}_2\bar{J}_2 + \bar{J}_1\bar{J}_3 + \hat{\omega}\bar{\nabla}\hat{\lambda}\right). \tag{2.60}$$

For the holomorphic one

$$\begin{aligned}\langle T\rangle =&\mathrm{STr}\left(\frac{1}{2}\langle J_2,J_2\rangle+\langle J_1,J_3\rangle-N\langle J_0\rangle\right)\\ =&\mathrm{STr}\left(\frac{1}{2}[\boldsymbol{N},T_m][\boldsymbol{N},T_n]\eta^{mn}-[\boldsymbol{N},T_\alpha][\boldsymbol{N},T_{\hat\alpha}]\eta^{\alpha\hat\alpha}\right.\\ &\left.+\frac{1}{2}\boldsymbol{N}\left(\{[\boldsymbol{N},T_{\hat\alpha}],T_\alpha\}\eta^{\alpha\hat\alpha}-\{[\boldsymbol{N},T_\alpha],T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}+[[\boldsymbol{N},T_m],T_n]\eta^{mn}\right)\right)\\ =&0.\end{aligned} \tag{2.61}$$

We used the results in (2.203,2.228) and the identity (2.96). A similar computations happens to the antiholomorphic $\bar{T}$, where now we use the results in (2.204,2.229) and the identity (2.97).

2.4.2 Conserved currents

The string sigma model is invariant under global left-multiplications by an element of $\mathfrak{psu}(2,2|4)$, $\delta g=\Lambda g$. We can calculate the conserved currents related to this symmetry using standard Noether method. The currents are given by

$$j=g\left(J_2+\frac{3}{2}J_3+\frac{1}{2}J_1-2N\right)g^{-1}=gAg^{-1}, \tag{2.62}$$

$$\bar{j}=g\left(\bar{J}_2+\frac{1}{2}\bar{J}_3+\frac{3}{2}\bar{J}_1-2\hat{N}\right)g^{-1}=g\bar{A}g^{-1}. \tag{2.63}$$

They should be free of divergences. To see that this is the case, it is easier to compute by parts:

$$\langle j\rangle=\langle g\rangle\boldsymbol{A}g_0^{-1}+\langle g,A\rangle g_0^{-1}+\langle g\boldsymbol{A},g^{-1}\rangle+g_0\langle A,g^{-1}\rangle+g_0\boldsymbol{A}\langle g^{-1}\rangle+g_0\langle A\rangle g_0^{-1}. \tag{2.64}$$

We have defined $\langle A\boldsymbol{B},C\rangle$ as usual, but taking $\boldsymbol{B}$ as a classical field, thus $\langle A\boldsymbol{B},C\rangle=$

$\langle A, \boldsymbol{B}C\rangle$. From (2.202) we get $\langle A\rangle = 0$, and using (2.201) we obtain

$$\begin{aligned}\langle g\rangle \boldsymbol{A} g_0^{-1} + \langle g\boldsymbol{A}, g^{-1}\rangle + g_0\boldsymbol{A}\langle g^{-1}\rangle =& \frac{1}{2} g_0 \langle [[\boldsymbol{A}, X]\,, X]\rangle g_0^{-1}\\ =& \frac{1}{2} g_0 \left([[\boldsymbol{A}, T_m]\,, T_n]\eta^{mn} + \{[\boldsymbol{A}, T_{\hat\alpha}], T_\alpha\}\eta^{\alpha\hat\alpha}\right.\\ & \left. - \{[\boldsymbol{A}, T_\alpha], T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}\right) g_0^{-1}. \end{aligned} \tag{2.65}$$

For the currents, using the results (2.207-2.212),

$$g_0^{-1}\langle g, J_1\rangle + \langle J_1, g^{-1}\rangle g_0 = - \{[\boldsymbol{J}_2, T_{\hat\alpha}], T_\alpha\}\eta^{\alpha\hat\alpha} - \{[\boldsymbol{J}_3, T_{\hat\alpha}], T_\alpha\}\eta^{\alpha\hat\alpha} + \{[\boldsymbol{N}, T_{\hat\alpha}], T_\alpha\}\eta^{\alpha\hat\alpha}, \tag{2.66}$$

$$g_0^{-1}\langle g, J_2\rangle + \langle J_2, g^{-1}\rangle g_0 = - [[\boldsymbol{J}_3, T_m], T_n]\eta^{mn} + [[\boldsymbol{N}, T_m], T_n]\eta^{mn}, \tag{2.67}$$

$$g_0^{-1}\langle g, J_3\rangle + \langle J_3, g^{-1}\rangle g_0 = - \{[\boldsymbol{N}, T_\alpha], T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}, \tag{2.68}$$

$$g_0^{-1}\langle g, N\rangle + \langle N, g^{-1}\rangle g_0 = 0, \tag{2.69}$$

but we already know that $\{[J_{1,3}, T_a]\,, T_b\}\, g^{ab} = 0$, for $a = \{i, m, \alpha, \hat\alpha\}$, see (2.98). Thus,

$$\begin{aligned} g_0^{-1}\langle j\rangle g_0 =& -\frac{1}{2}\left(\{[\boldsymbol{N}, T_{\hat\alpha}]\,, T_\alpha\} + \{[\boldsymbol{N}, T_\alpha]\,, T_{\hat\alpha}\}\right)\eta^{\alpha\hat\alpha}\\ &+ \frac{1}{2}\left(\{[\boldsymbol{J}_2, T_m]\,, T_n\}\,\eta^{mn} - \{[\boldsymbol{J}_2, T_\alpha]\,, T_{\hat\alpha}\}\,\eta^{\alpha\hat\alpha}\right). \end{aligned} \tag{2.70}$$

By lowering all the terms in the structure coefficients, we can see that the first term is just $(f_{i\alpha\hat\beta} f_{j\hat\alpha\beta} - f_{i\hat\alpha\beta} f_{j\alpha\hat\beta})\eta^{\alpha\hat\alpha}\eta^{\beta\hat\beta}$, and the second term is proportional to the dual coxeter number, see (2.96,2.97), which is 0. Thus, summing everything, we get

$$\langle j\rangle = 0. \tag{2.71}$$

For the antiholomorphic current we just obtain, using the same results as before,

$$g_0^{-1}\langle g, \bar J_1\rangle + \langle \bar J_1, g^{-1}\rangle g_0 = \{[\hat{\boldsymbol{N}}, T_{\hat\alpha}], T_\alpha\}\eta^{\alpha\hat\alpha}, \tag{2.72}$$

$$g_0^{-1}\langle g, \bar{J}_2\rangle + \langle \bar{J}_2, g^{-1}\rangle g_0 = -\,[[\bar{\boldsymbol{J}}_1, T_m], T_n]\eta^{mn} + [[\hat{\boldsymbol{N}}, T_m], T_n]\eta^{mn}, \tag{2.73}$$

$$g_0^{-1}\langle g, \bar{J}_3\rangle + \langle \bar{J}_3, g^{-1}\rangle g_0 = \{[\bar{\boldsymbol{J}}_1, T_\alpha], T_{\hat{\alpha}}\}\eta^{\alpha\hat{\alpha}} + \{[\bar{\boldsymbol{J}}_2, T_\alpha], T_{\hat{\alpha}}\}\eta^{\alpha\hat{\alpha}} - \{[\hat{\boldsymbol{N}}, T_\alpha], T_{\hat{\alpha}}\}\eta^{\alpha\hat{\alpha}}, \tag{2.74}$$

and using $\{[J_{1,3}, T_a], T_b\}\, g^{ab} = 0$ we see that doing the same as j, we arrive at $\langle \bar{j}\rangle = 0$.

2.4.3 b ghost

The pure spinor formalism does not have fundamental conformal ghosts. However, in a consistent string theory, the stress-energy tensor must be BRST exact $T = \{Q, b\}$. So there must exist a composite operator of ghost number -1 and conformal weight 2. The flat space b-ghost was first computed in [79] and a simplified expression for it in the $AdS_5 \times S^5$ background was derived in [80]. In our notation, the left and right moving b-ghosts can be written as

$$b = (\lambda\hat{\lambda})^{-1}\mathrm{STr}\left(\hat{\lambda}[J_2, J_3] + \{\omega, \hat{\lambda}\}[\lambda, J_1]\right) - \mathrm{STr}\left(\omega J_1\right), \tag{2.75}$$

$$\bar{b} = (\lambda\hat{\lambda})^{-1}\mathrm{STr}\left(\lambda[\bar{J}_2, \bar{J}_1] + \{\hat{\omega}, \lambda\}[\hat{\lambda}, \bar{J}_3]\right) - \mathrm{STr}\left(\hat{\omega}\bar{J}_3\right), \tag{2.76}$$

where $(\lambda\hat{\lambda}) = \lambda^A\hat{\lambda}^{\hat{A}}\eta_{A\hat{A}}$.

Let us first compute the divergent part of the left moving ghost; we will need the results from (2.244) to (2.254):

$$\begin{aligned}\langle b\rangle = &(\boldsymbol{\lambda}\hat{\boldsymbol{\lambda}})^{-1}\mathrm{STr}\langle\hat{\lambda}[J_2, J_3] + \{\omega, \hat{\lambda}\}[\lambda, J_1]\rangle - (\lambda\hat{\lambda})^{-2}\langle\lambda\hat{\lambda}\rangle\mathrm{STr}\left(\hat{\lambda}[J_2, J_3] + \{\omega, \hat{\lambda}\}[\lambda, J_1]\right)\\ &- (\boldsymbol{\lambda}\hat{\boldsymbol{\lambda}})^{-2}\langle(\lambda\hat{\lambda}), \mathrm{STr}\left(\hat{\lambda}[J_2, J_3] + \{\omega, \hat{\lambda}\}[\lambda, J_1]\right)\rangle - \mathrm{STr}\langle\omega J_1\rangle,\end{aligned} \tag{2.77}$$

The $\langle\lambda\hat{\lambda}\rangle$ term is easy,

$$\langle(\lambda\hat{\lambda})\rangle = -\,\boldsymbol{\lambda}^A\hat{\boldsymbol{\lambda}}^{\hat{A}} f_{Ai}^{B} f_{\hat{A}j}^{\hat{B}} g^{ij}\eta_{B\hat{B}} = 0; \tag{2.78}$$

where we have used (2.98). The $\langle \omega J_1 \rangle$ term is also 0. The other terms are

$$\begin{aligned}\mathrm{STr}\langle\hat{\lambda}[J_2,J_3]\rangle =& - \mathrm{STr}\left([\hat{\boldsymbol{\lambda}},T_i]([[\boldsymbol{J}_2,T_j],\boldsymbol{J}_3]+[\boldsymbol{J}_2,[\boldsymbol{J}_3,T_j]])g^{ij}\right)\\ =& - \mathrm{STr}\left([\hat{\boldsymbol{\lambda}},T_i][T_j,[\boldsymbol{J}_2,\boldsymbol{J}_3]]g^{ij}\right) = -\mathrm{STr}\left([\left[\hat{\boldsymbol{\lambda}},T_i\right],T_j][\boldsymbol{J}_2,\boldsymbol{J}_3]g^{ij}\right) = 0,\end{aligned} \tag{2.79}$$

we used $f_{i\alpha\hat{\beta}}f_{j\hat{\alpha}\beta}g^{ij}\eta^{\alpha\hat{\alpha}} = 0$, see (2.98). The next term is

$$\begin{aligned}\mathrm{STr}\langle\{\omega,\hat{\lambda}\}[\lambda,J_1]\rangle =& - \mathrm{STr}\Big(\{[\boldsymbol{\omega},T_i],[\hat{\boldsymbol{\lambda}},T_j]\}[\boldsymbol{\lambda},\boldsymbol{J}_1]+\{\boldsymbol{\omega},[\hat{\boldsymbol{\lambda}},T_i]\}[[\boldsymbol{\lambda},T_j],\boldsymbol{J}_1]\\ &+\{\boldsymbol{\omega},[\hat{\boldsymbol{\lambda}},T_i]\}[\boldsymbol{\lambda},[\boldsymbol{J}_1,T_j]]\Big)g^{ij}\\ =& - \mathrm{STr}\Big(\{\boldsymbol{\omega},[\hat{\boldsymbol{\lambda}},T_i]\}([T_j,[\lambda,\boldsymbol{J}_1]]+[[\boldsymbol{\lambda},T_j],\boldsymbol{J}_1]+[\boldsymbol{\lambda},[\boldsymbol{J}_1,T_j]])\Big)g^{ij}\\ =&0,\end{aligned} \tag{2.80}$$

which comes from the Jacobi identity, see appendix B. The remaining terms are computed using

$$\lambda^A[\hat{\lambda}^{\hat{A}},T_i]\eta_{A\hat{A}} = - [\lambda^A,T_i]\hat{\lambda}^{\hat{A}}\eta_{A\hat{A}} = \{\lambda,\hat{\lambda}\}^j g_{ij}, \tag{2.81}$$

thus

$$\langle(\lambda\hat{\lambda}),\mathrm{STr}\left(\hat{\lambda}[J_2,J_3]\right)\rangle = \mathrm{STr}\left([\hat{\boldsymbol{\lambda}},\{\boldsymbol{\lambda},\hat{\boldsymbol{\lambda}}\}][\boldsymbol{J}_2,\boldsymbol{J}_3]\right)+\mathrm{STr}\left(\hat{\boldsymbol{\lambda}}[[\boldsymbol{J}_2,\boldsymbol{J}_3],\{\boldsymbol{\lambda},\hat{\boldsymbol{\lambda}}\}]\right)=0, \tag{2.82}$$

$$\begin{aligned}\langle(\lambda\hat{\lambda}),\mathrm{STr}\left(\{\omega,\hat{\lambda}\}[\lambda,J_1]\right)\rangle =& \mathrm{STr}\Big(\{\boldsymbol{\omega},[\hat{\boldsymbol{\lambda}},\{\boldsymbol{\lambda},\hat{\boldsymbol{\lambda}}\}]\}[\boldsymbol{\lambda},\boldsymbol{J}_1]-\{[\boldsymbol{\omega},\{\boldsymbol{\lambda},\hat{\boldsymbol{\lambda}}\}],\hat{\boldsymbol{\lambda}}\}[\boldsymbol{\lambda},\boldsymbol{J}_1]\\ &+[\{\boldsymbol{\omega},\hat{\boldsymbol{\lambda}}\},\{\boldsymbol{\lambda},\hat{\boldsymbol{\lambda}}\}][\boldsymbol{\lambda},\boldsymbol{J}_1]\Big)\\ =&2\mathrm{STr}\left(\{\boldsymbol{\omega},[\hat{\boldsymbol{\lambda}},\{\boldsymbol{\lambda},\hat{\boldsymbol{\lambda}}\}]\}[\boldsymbol{\lambda},\boldsymbol{J}_1]\right)=0,\end{aligned} \tag{2.83}$$

which is true due to the pure spinor condition.

For $\langle \bar{b} \rangle$ one needs to use the same relations from above.

2.5 Conclusions and further directions

In this paper we outlined a general method to compute the logarithmic divergences of local operators of the pure spinor string in an $AdS_5 \times S^5$ background. In the text we derived in detail the case for operators up to classical dimension two, but the method extends to any classical dimension. Although the worldsheet anomalous dimension is not related to a physical observable, as in the case of N=4 SYM, physical vertex operators should not have quantum corrections to their classical dimension. The main application of our work is to systematize the search for physical vertex operators. We presented some consistency checks verifying that some conserved local operators are not renormalized.

The basic example is the radius operator discussed in [80]. It has ghost number $(1,1)$ and zero classical dimension. In our notation it can be written as

$$V = \mathrm{Str}(\lambda\hat{\lambda}), \tag{2.84}$$

If we apply the pairing rules to compute $\langle V \rangle$ we obtain

$$\langle V \rangle = -\mathrm{I} g^{ij} \mathrm{Str}([\lambda, T_i][\hat{\lambda}, T_j]) = 0, \tag{2.85}$$

where in the last equality we replaced the structure constants and used one of the identities in the Appendix A. This can be generalized to other massless and massive vertex operators. We plan to return to this problem in the future.

A more interesting direction is to try to organize the dilatation operator including the higher derivative contributions. As we commented in the introduction, the difficulty here is that the pure spinor action is not an usual coset action as in [74, 75]. However, it might still be possible to obtain the complete one loop dilatation operator restricting to some subsector of the $\mathfrak{psu}(2,2|4)$ algebra, in a way similar as it was done for super Yang-Mills dilatation operator [50].

2.A Notation and conventions

Here we collect the conventions and notation used in this paper. We work with euclidean world sheet with coordinates $(z, \bar{z})$.

We split the current as $J = A + K$. We define $K = J_1 + J_2 + J_3 \in \mathfrak{psu}(2,2|4)$ and $A = J_0$ belongs to the stability group algebra.[3] The notation that we use for the different graded generators is given by

$$J_0 = J_0^i T_i \quad ; \quad J_1 = J_1^\alpha T_\alpha \quad ; \quad J_2 = J_2^m T_m \quad ; \quad J_3 = J_3^{\hat{\alpha}} T_{\hat{\alpha}}. \tag{2.86}$$

The ghosts fields are defined as

$$\lambda = \lambda^A T_A \quad ; \quad \omega = -\omega_A \eta^{A\hat{A}} T_{\hat{A}} \quad ; \quad \hat{\lambda} = \hat{\lambda}^{\hat{A}} T_{\hat{A}} \quad ; \quad \hat{\omega} = \hat{\omega}_{\hat{B}} \eta^{B\hat{B}} T_B. \tag{2.87}$$

The only non-zero Str of generators are

$$g_{ij} = \mathrm{STr} T_i T_j, \tag{2.88}$$

$$\eta_{mn} = \mathrm{STr} T_m T_n, \tag{2.89}$$

$$\eta_{\alpha\hat{\alpha}} = \mathrm{STr} T_\alpha T_{\hat{\alpha}}. \tag{2.90}$$

For the raising and lowering of fermionic indices in the structure constants we use

$$f_{m\alpha\beta} = \eta_{\alpha\hat{\alpha}} f^{\hat{\alpha}}_{\beta m} \qquad \text{and} \qquad f_{m\hat{\alpha}\hat{\beta}} = -\eta_{\alpha\hat{\alpha}} f^{\alpha}_{\hat{\beta} m}, \tag{2.91}$$

[3] Although we did not use the K term in the main text, it will be useful from now on to use this term in order to pack several results.

and for the $f_{\alpha\hat{\alpha}i}$ the rule is the same. For the bosonic case we use the standard raising/lowering procedure.

2.B Some identities for $\mathfrak{psu}(2,2|4)$

Let A, B and C be bosons, X, Y and Z fermions, then, the generalized Jacobi Identities are

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0, \tag{2.92}$$

$$[A,[B,X]]+[B,[X,A]]+[X,[A,B]]=0, \tag{2.93}$$

$$\{X,[Y,A]\}+\{Y,[X,A]\}+[A,\{X,Y\}]=0, \tag{2.94}$$

$$[X,\{Y,Z\}]+[Y,\{Z,X\}]+[Z,\{X,Y\}]=0. \tag{2.95}$$

In this theory the dual-coxeter number is 0, this implies

$$[[A,T_i],T_j]g^{ij}-\{[A,T_\alpha],T_{\hat{\alpha}}\}\eta^{\alpha\hat{\alpha}}+[[A,T_m],T_n]\eta^{mn}+\{[A,T_{\hat{\alpha}}],T_\alpha\}\eta^{\alpha\hat{\alpha}}=0, \tag{2.96}$$

$$[[X,T_i],T_j]g^{ij}-[\{X,T_\alpha\},T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}}+[[X,T_m],T_n]\eta^{mn}+[\{X,T_{\hat{\alpha}}\},T_\alpha]\eta^{\alpha\hat{\alpha}}=0. \tag{2.97}$$

The Jacobi identity yields $f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}\eta^{mn}\eta^{\alpha\hat{\alpha}}=0$ and $f_{i\alpha\hat{\beta}}f_{j\hat{\alpha}\beta}g^{ij}\eta^{\alpha\hat{\alpha}}=0$. This implies that

$$[[J_{1,3},T_i],T_j]g^{ij}=[[J_{1,3},T_n],T_m]\eta^{mn}=\{[J_{1,3},T_\alpha],T_{\hat{\alpha}}\}\eta^{\alpha\hat{\alpha}}=\{[J_{1,3},T_{\hat{\alpha}}],T_\alpha\}\eta^{\alpha\hat{\alpha}}=0, \tag{2.98}$$

$$[[\omega+\lambda+\hat{\omega}+\hat{\lambda},T_i],T_j]g^{ij}=[[\omega+\lambda+\hat{\omega}+\hat{\lambda},T_n],T_m]\eta^{mn}=0, \tag{2.99}$$

$$[\{\omega+\lambda+\hat{\omega}+\hat{\lambda},T_\alpha\},T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}}=[\{\omega+\lambda+\hat{\omega}+\hat{\lambda},T_{\hat{\alpha}}\},T_\alpha]\eta^{\alpha\hat{\alpha}}=0. \tag{2.100}$$

Another useful property of this theory is the pure spinor condition Eq. 2.41. Using it, it is easy to prove that

$$\left[\hat{\lambda}, \left[\hat{\lambda}, A\right]_{\pm}\right]_{\mp} = \left[\lambda, [\lambda, A]_{\pm}\right]_{\mp} = 0. \tag{2.101}$$

2.C Complete solution of the SD equation for the $AdS_5 \times S^5$ pure spinor string

In this Appendix we apply the method explained in Section 2, and generalized in Section 3, to the $AdS_5 \times S^5$ superstring. Step by step, the procedure is as follows:

1. Using an expansion around a classical background, $g = g_0 e^X$, we compute all the currents up to second order in X,

2. Expand the action (2.37) up to second order in X,

3. Write down the Schwinger-Dyson equation for the model and compute the interaction matrix,

4. Compute the Green functions in powers of $\frac{1}{k}$,

5. Compute $\langle \phi^i, \phi^j \rangle$.

The expansion of the currents was already done in (2.42). The remaining subsections are devoted, each one, to each of the steps listed above.

We will drop the use of the boldface notation for the background fields in this section. All the quantum corrections come from either an x-term or a $\left(\delta\omega, \delta\lambda, \delta\hat{\omega}, \delta\hat{\lambda}\right)$ -term. Thus, every field in S_{int}, the F-terms, the Green's functions and in the RHS of the pairing rules should be treated as classical.

2.C.1 Action

In (2.43) we showed the kinetic part of the expansion of (2.37) and we promised to show the interaction part later, here we fulfil our promise. Up to second order in X the interaction part is

$$
\begin{aligned}
S_{int} =&\frac{R^2}{2\pi}\int d^2z\,\Big[\frac{1}{2}\bar\partial x_1^\alpha x_1^\beta J_2^m f_{m\alpha\beta}+\frac{1}{2}x_1^\alpha x_1^\beta J_3^{\hat\alpha}\bar J_3^{\hat\beta}f_{i\alpha\hat\alpha}f_{j\beta\hat\beta}g^{ij}+\frac{1}{8}\left(3x_1^\alpha\bar\partial x_2^m-5\bar\partial x_1^\alpha x_2^m\right)J_1^\beta f_{m\alpha\beta}\\
&+\frac{1}{8}x_1^\alpha x_2^m\Big(-\partial\bar J_1^\beta f_{m\alpha\beta}+\left[3J_2^n\bar J_3^{\hat\alpha}+5\bar J_2^n J_3^{\hat\alpha}\right]f_{i\hat\alpha\alpha}f_{jmn}g^{ij}+3\left[J_2^n\bar J_3^{\hat\alpha}-\bar J_2^n J_3^{\hat\alpha}\right]f_{n\alpha\beta}f_{m\hat\beta\hat\alpha}\eta^{\beta\hat\beta}\Big)\\
&-\frac{1}{4}x_1^\alpha x_3^{\hat\alpha}\Big(\left[\bar J_1^\beta J_3^{\hat\beta}-J_1^\beta\bar J_3^{\hat\beta}\right]f_{m\alpha\beta}f_{n\hat\alpha\hat\beta}\eta^{mn}+\left[J_1^\beta\bar J_3^{\hat\beta}+3\bar J_1^\beta J_3^{\hat\beta}\right]f_{i\hat\alpha\beta}f_{j\alpha\hat\beta}g^{ij}\\
&+J_2^m\bar J_2^n\left[f_{m\alpha\beta}f_{n\hat\alpha\hat\beta}-f_{n\alpha\beta}f_{m\hat\alpha\hat\beta}\right]\eta^{\beta\hat\beta}\Big)+\frac{1}{2}\partial x_3^{\hat\alpha}x_3^{\hat\beta}\bar J_2^m f_{m\hat\alpha\hat\beta}+\frac{1}{2}x_3^{\hat\alpha}x_3^{\hat\beta}J_1^\alpha\bar J_1^\beta f_{i\alpha\hat\alpha}f_{j\beta\hat\beta}g^{ij}\\
&-\frac{1}{2}x_2^m x_2^n\Big(\left[J_1^\alpha\bar J_3^{\hat\alpha}-\bar J_1^\alpha J_3^{\hat\alpha}\right]f_{m\alpha\beta}f_{n\hat\alpha\hat\beta}\eta^{\beta\hat\beta}+J_2^p\bar J_2^q f_{ipm}f_{jqn}g^{ij}\Big)+\frac{1}{8}\left(3\partial x_2^m x_3^{\hat\alpha}\right.\\
&\left.-5x_2^m\partial x_3^{\hat\alpha}\right)\bar J_3^{\hat\beta}f_{m\hat\alpha\hat\beta}+\frac{1}{8}x_2^m x_3^{\hat\alpha}\Big(-\bar\partial J_3^{\hat\beta}f_{m\hat\alpha\hat\beta}\\
&+3\left[\bar J_1^\alpha J_2^n-J_1^\alpha\bar J_2^n\right]f_{m\alpha\beta}f_{n\hat\alpha\hat\beta}\eta^{\beta\hat\beta}+\left[3J_1^\alpha\bar J_2^n+5\bar J_1^\alpha J_2^n\right]f_{i\alpha\hat\alpha}f_{jmn}g^{ij}\Big)\\
&-\delta^2(N^i\hat N^j)g_{ij}-x_1^\alpha\left(\delta N^i\bar J_3^{\hat\alpha}+\delta\hat N^i J_3^{\hat\alpha}\right)f_{i\alpha\hat\alpha}+x_2^m\left(\delta N^i\bar J_2^n+\delta\hat N^i J_2^n\right)f_{imn}\\
&-x_3^{\hat\alpha}\left(\delta N^i\bar J_1^\alpha+\delta\hat N^i J_1^\alpha\right)f_{i\alpha\hat\alpha}-\frac{1}{2}x_1^\alpha x_1^\beta\left(N^i\bar J_2^m+\hat N^i J_2^m\right)f_{m\alpha\mu}f_{i\beta\hat\mu}\eta^{\mu\hat\mu}\\
&-\frac{1}{2}x_1^\alpha x_2^m\left(N^i\bar J_1^\beta+\hat N^i J_1^\beta\right)\left(f_{ipm}f_{q\alpha\beta}\eta^{pq}+f_{i\alpha\hat\mu}f_{m\beta\mu}\eta^{\mu\hat\mu}\right)\\
&+\frac{1}{2}\left(\partial x_1^\alpha x_3^{\hat\alpha}-x_1^\alpha\partial x_3^{\hat\alpha}\right)\hat N^i f_{i\alpha\hat\alpha}+\frac{1}{2}x_2^m\left(\bar\partial x_2^n N^i+\partial x_2^n\hat N^i\right)f_{imn}\\
&-\frac{1}{2}x_2^m x_3^{\hat\alpha}\left(N^i\bar J_3^{\hat\beta}+\hat N^i J_3^{\hat\beta}\right)\left(f_{ipm}f_{q\hat\alpha\hat\beta}\eta^{pq}-f_{i\hat\alpha\mu}f_{m\hat\beta\hat\mu}\eta^{\mu\hat\mu}\right)\\
&+\frac{1}{2}x_3^{\hat\alpha}x_3^{\hat\beta}\left(N^i\bar J_2^m+\hat N^i J_2^m\right)f_{m\hat\alpha\hat\mu}f_{i\mu\hat\beta}\eta^{\mu\hat\mu}+\frac{1}{2}\left(\bar\partial x_1^\alpha x_3^{\hat\alpha}-x_1^\alpha\bar\partial x_3^{\hat\alpha}\right)N^i f_{i\alpha\hat\alpha}\Big],
\end{aligned}\tag{2.102}
$$

with

$$N^i =-\omega_A\lambda^B\eta^{A\hat B}f^i_{B\hat B},\tag{2.103}$$

$$\hat N^i =\hat\omega_{\hat A}\hat\lambda^{\hat B}\eta^{A\hat A}f^i_{B\hat B},\tag{2.104}$$

$$\delta N^i =(\delta\omega_A\lambda^B+\omega_A\delta\lambda^B)\eta^{A\hat B}f^i_{B\hat B},\tag{2.105}$$

$$\delta\hat{N}^i = (\delta\hat{\omega}_{\hat{A}}\hat{\lambda}^{\hat{B}} + \hat{\omega}_{\hat{A}}\delta\hat{\lambda}^{\hat{B}})\eta^{A\hat{A}}f^i_{B\hat{B}}, \tag{2.106}$$

$$\delta^2(N^i\hat{N}^j) = \delta N^i\delta\hat{N}^j - \delta\omega_A\delta\lambda^B\eta^{A\hat{B}}f^i_{B\hat{B}}\hat{N}^j + N^i\delta\hat{\omega}_{\hat{A}}\delta\hat{\lambda}^{\hat{B}}\eta^{B\hat{A}}f^j_{B\hat{B}}. \tag{2.107}$$

The lack of covariant derivatives is, as explained previously, because the pure spinor sigma model is anomaly free. This means that physical quantities only appear in gauge invariant expressions, thus the interchange $\partial \leftrightarrow \nabla$ can be done at any moment in our computation. A more detailed explanation can be found in Subsection 3.2.

2.C.2 Schwinger-Dyson equation and the Interaction Matrix

The Schwinger-Dyson equation in momentum space for (2.37) reads

$$G^{\alpha\Lambda} = \frac{2\pi}{R^2}\frac{\eta^{\alpha\Lambda}}{|k|^2} + \frac{1}{|k|^2}(ik\bar{\partial} + i\bar{k}\partial + \Box)G^{\alpha\Lambda} - \frac{\eta^{\alpha\Omega}}{|k|^2}F_{\Sigma\Omega}G^{\Sigma\Lambda}, \tag{2.108}$$

$$G^{m\Lambda} = \frac{2\pi}{R^2}\frac{\eta^{m\Lambda}}{|k|^2} + \frac{1}{|k|^2}(ik\bar{\partial} + i\bar{k}\partial + \Box)G^{\alpha\Lambda} - \frac{\eta^{m\Omega}}{|k|^2}F_{\Sigma\Omega}G^{\Sigma\Lambda}, \tag{2.109}$$

$$G^{\hat{\alpha}\Lambda} = -\frac{2\pi}{R^2}\frac{\eta^{\hat{\alpha}\Lambda}}{|k|^2} + \frac{1}{|k|^2}(ik\bar{\partial} + i\bar{k}\partial + \Box)G^{\alpha\Lambda} + \frac{\eta^{\Omega\hat{\alpha}}}{|k|^2}F_{\Sigma\Omega}G^{\Sigma\Lambda}, \tag{2.110}$$

$$G_A{}^\Lambda = \frac{2\pi}{R^2}\frac{i}{\bar{k}}\delta_A^\Lambda + \frac{i}{\bar{k}}\bar{\partial}G_A{}^\Lambda - \frac{i}{\bar{k}}F_{\Sigma A}G^{\Sigma\Lambda}, \tag{2.111}$$

$$G^{B\Lambda} = -\frac{2\pi}{R^2}\frac{i}{\bar{k}}\delta_{B\Lambda} + \frac{i}{\bar{k}}\bar{\partial}G^{B\Lambda} + \frac{i}{\bar{k}}F_\Sigma{}^BG^{\Sigma\Lambda}, \tag{2.112}$$

$$G_{\hat{A}}{}^\Lambda = \frac{2\pi}{R^2}\frac{i}{k}\delta_{\hat{A}}^\Lambda + \frac{i}{k}\partial G_{\hat{A}}{}^\Lambda - \frac{i}{k}F_{\Sigma\hat{A}}G^{\Sigma\Lambda}, \tag{2.113}$$

$$G^{\hat{B}\Lambda} = -\frac{2\pi}{R^2}\frac{i}{k}\delta_{\hat{B}\Lambda} + \frac{i}{k}\partial G^{\hat{B}\Lambda} + \frac{i}{k}F_\Sigma{}^{\hat{B}}G^{\Sigma\Lambda}, \tag{2.114}$$

where $\Lambda = \{\alpha, m, \hat{\alpha}, {}^A, {}_A, {}^{\hat{A}}, {}_{\hat{A}}\}$.

The interaction matrix is given by

$$F_{\Sigma\Omega}(x,y) = \frac{\overleftarrow{\delta}}{\overleftarrow{\delta}\,\Phi^\Sigma(y)}\frac{\delta S_{int}}{\delta\Phi^\Omega(x)}. \tag{2.115}$$

The directional derivative means that we compute the functional derivative of S_{int} with respect to Φ^{Σ} acting from right to left. Because we are working in momentum space is useful to write also F in momentum space, for that reason the equation we work with is

$$F_{\Lambda\Omega}(x,k)f(x) = \int d^2y \frac{\overleftarrow{\delta}}{\overleftarrow{\delta}\,\Phi^{\Sigma}(y)} \frac{\delta S_{int}}{\delta\Phi^{\Omega}(x)} \exp(iky)f(y). \tag{2.116}$$

Note that the $f(y)$ stands for the previous Green's function and the exponential came from the Fourier Transform. The directional derivative has the same meaning as above.

We organize the interaction matrix by the $\mathbb{Z}_4$ charge of its indices, and in the end we add the ghosts contributions.

The first we compute the $F_{\alpha\Lambda}$ terms of the matrix:

$$\begin{aligned}
F_{\alpha\beta} =& - J_2^m(i\bar{k}+\bar{\partial})f_{m\alpha\beta} - \frac{1}{2}\bar{\partial}J_2^m f_{m\alpha\beta} - \frac{1}{2}J_3^{\hat{\alpha}}\bar{J}_3^{\hat{\beta}}\left(f_{i\alpha\hat{\alpha}}f_{j\beta\hat{\beta}} - f_{i\beta\hat{\alpha}}f_{j\alpha\hat{\beta}}\right)g^{ij} \\
&+ \frac{1}{2}\left(N^i\bar{J}_2^m + \hat{N}^i J_2^m\right)\left(f_{m\alpha\mu}f_{i\beta\hat{\mu}} - f_{m\beta\mu}f_{i\alpha\hat{\mu}}\right)\eta^{\mu\hat{\mu}}, && (2.117)\\
F_{\alpha m} =& J_1^{\beta}\left(i\bar{k}+\bar{\partial}\right)f_{m\alpha\beta} + \frac{1}{8}\left(\partial\bar{J}_1^{\beta} + 3\bar{\partial}J_1^{\beta}\right)f_{m\alpha\beta} - \frac{1}{8}\left(3J_2^n\bar{J}_3^{\hat{\alpha}} + 5\bar{J}_2^n J_3^{\hat{\alpha}}\right)f_{i\hat{\alpha}\alpha}f_{jmn}g^{ij} && (2.118)\\
&- \frac{3}{8}\left(J_2^n\bar{J}_3^{\hat{\alpha}} - \bar{J}_2^n J_3^{\hat{\alpha}}\right)f_{n\alpha\beta}f_{m\hat{\beta}\hat{\alpha}}\eta^{\beta\hat{\beta}} + \frac{1}{2}\left(N^i\bar{J}_1^{\beta} - \hat{N}^i J_1^{\beta}\right)\left(f_{ipm}f_{q\alpha\beta}\eta^{pq} + f_{i\alpha\hat{\mu}}f_{m\beta\mu}\eta^{\mu\hat{\mu}}\right)\\
F_{\alpha\hat{\alpha}} =& - N^i f_{i\alpha\hat{\alpha}}\left(i\bar{k}+\bar{\partial}\right) - \hat{N}^i f_{i\alpha\hat{\alpha}}\left(ik+\partial\right) + \frac{1}{4}\left(\bar{J}_1^{\beta}J_3^{\hat{\beta}} - J_1^{\beta}\bar{J}_3^{\hat{\beta}}\right)f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}\eta^{mn}\\
&+ \frac{1}{4}\left(J_1^{\beta}\bar{J}_3^{\hat{\beta}} + 3\bar{J}_1^{\beta}J_3^{\hat{\beta}}\right)f_{i\hat{\alpha}\beta}f_{j\alpha\hat{\beta}}g^{ij} + \frac{1}{4}J_2^m\bar{J}_2^n\left(f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}} - f_{n\alpha\beta}f_{m\hat{\alpha}\hat{\beta}}\right)\eta^{\beta\hat{\beta}}, && (2.119)\\
F_{\alpha B} =& - \omega_A\bar{J}_3^{\hat{\alpha}}A_{B\ \alpha\hat{\alpha}}^{A} = -F_{B\alpha}, && (2.120)\\
F_{\alpha}{}^{A} =& - \lambda^B\bar{J}_3^{\hat{\alpha}}A_{B\ \alpha\hat{\alpha}}^{A} = -F^{A}_{\ \alpha}, && (2.121)\\
F_{\alpha\hat{B}} =& \hat{\omega}_{\hat{A}}J_3^{\hat{\alpha}}A_{B\ \alpha\hat{\alpha}}^{A} = -F_{\hat{B}\alpha}, && (2.122)\\
F_{\alpha}{}^{\hat{A}} =& \hat{\lambda}^{\hat{B}}J_3^{\hat{\alpha}}A_{B\ \alpha\hat{\alpha}}^{A} = -F^{\hat{A}}_{\ \alpha}. && (2.123)
\end{aligned}$$

The terms of the $F_{m\Lambda}$ kind are

$$F_{m\alpha} = J_1^{\beta}\left(i\bar{k}+\bar{\partial}\right)f_{m\alpha\beta} + \frac{1}{2}\left(N^i\bar{J}_1^{\beta} + \hat{N}^i J_1^{\beta}\right)\left(f_{ipm}f_{q\alpha\beta}\eta^{pq} + f_{i\alpha\hat{\mu}}f_{m\beta\mu}\eta^{\mu\hat{\mu}}\right)$$

$$
\begin{aligned}
&+\frac{1}{8}\left(3J_2^n\bar{J}_3^{\hat{\alpha}}+5\bar{J}_2^nJ_3^{\hat{\alpha}}\right)f_{i\hat{\alpha}\alpha}f_{jmn}g^{ij}+\frac{3}{8}\left(J_2^n\bar{J}_3^{\hat{\alpha}}-\bar{J}_2^nJ_3^{\hat{\alpha}}\right)f_{n\alpha\beta}f_{m\hat{\beta}\hat{\alpha}}\eta^{\beta\hat{\beta}}\\
&+\frac{1}{8}\left(5\bar{\partial}J_1^{\beta}-\partial\bar{J}_1^{\beta}\right)f_{m\alpha\beta},
\end{aligned}
\tag{2.124}
$$

$$
\begin{aligned}
F_{mn}=&N^if_{imn}\left(i\bar{k}+\bar{\partial}\right)+\hat{N}^if_{imn}\left(ik+\partial\right)\\
&-\frac{1}{2}\left(J_1^{\alpha}\bar{J}_3^{\hat{\alpha}}-\bar{J}_1^{\alpha}J_3^{\hat{\alpha}}\right)(f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}+f_{n\alpha\beta}f_{m\hat{\alpha}\hat{\beta}})\eta^{\beta\hat{\beta}}-\frac{1}{2}J_2^p\bar{J}_2^q(f_{ipm}f_{jqn}+f_{ipn}f_{jqm})g^{ij},
\end{aligned}
\tag{2.125}
$$

$$
\begin{aligned}
F_{m\hat{\alpha}}=&\bar{J}_3^{\hat{\beta}}f_{m\hat{\alpha}\hat{\beta}}\left(ik+\partial\right)+\frac{1}{8}\left(5\partial\bar{J}_3^{\hat{\beta}}-\bar{\partial}J_3^{\hat{\beta}}\right)f_{m\hat{\alpha}\hat{\beta}}+\frac{3}{8}\left(\bar{J}_1^{\alpha}J_2^n-J_1^{\alpha}\bar{J}_2^n\right)f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}\eta^{\beta\hat{\beta}}\\
&+\frac{1}{8}\left(3J_1^{\alpha}\bar{J}_2^n+5\bar{J}_1^{\alpha}J_2^n\right)f_{i\alpha\hat{\alpha}}f_{jmn}g^{ij}+\frac{1}{2}\left(N^i\bar{J}_3^{\hat{\beta}}+\hat{N}^iJ_3^{\hat{\beta}}\right)\left(f_{ipm}f_{q\hat{\alpha}\hat{\beta}}\eta^{pq}-f_{i\hat{\alpha}\mu}f_{m\hat{\beta}\hat{\mu}}\eta^{\mu\hat{\mu}}\right),
\end{aligned}
\tag{2.126}
$$

$$
F_{mB}=-\omega_A\bar{J}_2^nA^A_{B\ mn}=F_{Bm},
\tag{2.127}
$$

$$
F_m{}^A=-\lambda^B\bar{J}_2^nA^A_{B\ mn}=F^B{}_m,
\tag{2.128}
$$

$$
F_{m\hat{B}}=\hat{\omega}_{\hat{A}}J_2^nA^A_{B\ mn}=F_{\hat{B}m},
\tag{2.129}
$$

$$
F_m{}^{\hat{A}}=\lambda^{\hat{B}}J_2^nA^A_{B\ mn}=F^{\hat{A}}{}_m.
\tag{2.130}
$$

The last contribution from the non-ghost terms is given by the $F_{\hat{\alpha}\Lambda}$ elements:

$$
\begin{aligned}
F_{\hat{\alpha}\alpha}=&-N^if_{i\alpha\hat{\alpha}}\left(i\bar{k}+\bar{\partial}\right)-\hat{N}^if_{i\alpha\hat{\alpha}}\left(ik+\partial\right)-\frac{1}{4}\left(J_1^{\beta}J_3^{\hat{\beta}}-J_1^{\beta}\bar{J}_3^{\hat{\beta}}\right)f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}\eta^{mn}\\
&-\frac{1}{4}\left(J_1^{\beta}\bar{J}_3^{\hat{\beta}}+3\bar{J}_1^{\beta}J_3^{\hat{\beta}}\right)f_{i\hat{\alpha}\beta}f_{j\alpha\hat{\beta}}g^{ij}-\frac{1}{4}J_2^m\bar{J}_2^n\left(f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}-f_{n\alpha\beta}f_{m\hat{\alpha}\hat{\beta}}\right)\eta^{\beta\hat{\beta}},
\end{aligned}
\tag{2.131}
$$

$$
\begin{aligned}
F_{\hat{\alpha}m}=&\bar{J}_3^{\hat{\beta}}f_{m\hat{\alpha}\hat{\beta}}\left(ik+\partial\right)+\frac{1}{8}\left(3\partial\bar{J}_3^{\hat{\beta}}+\bar{\partial}J_3^{\hat{\beta}}\right)f_{m\hat{\alpha}\hat{\beta}}-\frac{3}{8}\left(\bar{J}_1^{\alpha}J_2^n-J_1^{\alpha}\bar{J}_2^n\right)f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}\eta^{\beta\hat{\beta}}\\
&-\frac{1}{8}\left(3J_1^{\alpha}\bar{J}_2^n+5\bar{J}_1^{\alpha}J_2^n\right)f_{i\alpha\hat{\alpha}}f_{jmn}g^{ij}-\frac{1}{2}\left(N^i\bar{J}_3^{\hat{\beta}}+\hat{N}^iJ_3^{\hat{\beta}}\right)\left(f_{ipm}f_{q\hat{\alpha}\hat{\beta}}\eta^{pq}-f_{i\hat{\alpha}\mu}f_{m\hat{\beta}\hat{\mu}}\eta^{\mu\hat{\mu}}\right),
\end{aligned}
\tag{2.132}
$$

$$
\begin{aligned}
F_{\hat{\alpha}\hat{\beta}}=&-\bar{J}_2^m\left(ik+\partial\right)f_{m\hat{\alpha}\hat{\beta}}-\frac{1}{2}\partial\bar{J}_2^mf_{m\hat{\alpha}\hat{\beta}}-\frac{1}{2}J_1^{\alpha}\bar{J}_1^{\beta}\left(f_{i\alpha\hat{\alpha}}f_{j\beta\hat{\beta}}-f_{i\beta\hat{\alpha}}f_{j\alpha\hat{\beta}}\right)g^{ij}\\
&-\frac{1}{2}\left(N^i\bar{J}_2^m+\hat{N}^iJ_2^m\right)\left(f_{m\hat{\alpha}\hat{\mu}}f_{i\mu\hat{\beta}}-f_{m\hat{\beta}\hat{\mu}}f_{i\mu\hat{\alpha}}\right)\eta^{\mu\hat{\mu}},
\end{aligned}
\tag{2.133}
$$

$$
F_{\hat{\alpha}B}=-\omega_A\bar{J}_1^{\alpha}A^A_{B\ \alpha\hat{\alpha}}=-F_{B\hat{\alpha}},
\tag{2.134}
$$

$$
F_{\hat{\alpha}}{}^A=-\lambda^B\bar{J}_1^{\alpha}A^A_{B\ \alpha\hat{\alpha}}=-F^A{}_{\hat{\alpha}},
\tag{2.135}
$$

$$
F_{\hat{\alpha}\hat{B}}=\hat{\omega}_{\hat{A}}J_1^{\alpha}A^A_{B\ \alpha\hat{\alpha}}=-F_{\hat{B}\hat{\alpha}},
\tag{2.136}
$$

$$
F_{\hat{\alpha}B}=\lambda^{\hat{A}}J_1^{\alpha}A^A_{B\ \alpha\hat{\alpha}}=-F^{\hat{B}}{}_{\hat{\alpha}}.
\tag{2.137}
$$

Finally we compute the pure ghost terms, and we save some trees by not adding the symmetric terms already listed:

$$F_B{}^A = \hat{N}_A^B = F^A{}_B, \tag{2.138}$$
$$F_B{}^{\hat{A}} = \omega_A \hat{\lambda}^{\hat{B}} A^{A\hat{A}}_{B\hat{B}} = F^{\hat{A}}{}_B, \tag{2.139}$$
$$F_{B\hat{B}} = \omega_A \hat{\omega}_{\hat{A}} A^{A\hat{A}}_{B\hat{B}} = F_{B\hat{A}}, \tag{2.140}$$
$$F^A{}_{\hat{B}} = \lambda^B \hat{\omega}_{\hat{A}} A^{A\hat{A}}_{B\hat{B}} = F_{\hat{B}}{}^A, \tag{2.141}$$
$$F^{A\hat{A}} = \lambda^B \hat{\lambda}^{\hat{B}} A^{A\hat{A}}_{B\hat{B}} = F^{\hat{A}A}, \tag{2.142}$$
$$F_{\hat{B}}{}^{\hat{A}} = N^{\hat{A}}_{\hat{B}} = F^{\hat{A}}{}_{\hat{B}}, \tag{2.143}$$

where we have defined

$$A^{A\hat{A}}_{B\hat{B}} = \eta^{A\hat{C}} \eta^{C\hat{A}} f^i_{B\hat{C}} f^j_{\hat{B}C} g_{ij}, \tag{2.144}$$
$$\hat{N}^B_A = \hat{\omega}_{\hat{A}} \hat{\lambda}^{\hat{B}} A^{A\hat{A}}_{B\hat{B}}, \tag{2.145}$$
$$N^{\hat{B}}_{\hat{A}} = \omega_A \lambda^B A^{A\hat{A}}_{B\hat{B}}. \tag{2.146}$$

2.C.3 Green functions

With all the previous results, we begin the computation of the Green's Functions as a power series in $1/k$. We follow the prescription given in (2.52). The Green functions are presented order by order, which makes the reading easier.

The only contributions of order $1/k$ come from the ghosts propagators

$$G_{1A}{}^B = \frac{2\pi}{R^2} \frac{i}{k} \delta^B_A = -G_1^B{}_A, \tag{2.147}$$
$$G_{1\hat{A}}{}^{\hat{B}} = \frac{2\pi}{R^2} \frac{i}{k} \delta^{\hat{B}}_{\hat{A}} = -G_1^{\hat{B}}{}_{\hat{A}}. \tag{2.148}$$

For the $1/k^2$ terms, we have a contribution from the non-ghosts propagators

$$G_2^{\alpha\hat{\alpha}} = \frac{2\pi}{R^2}\frac{1}{|k|^2}\eta^{\alpha\hat{\alpha}} = -G_2^{\hat{\alpha}\alpha}, \tag{2.149}$$
$$G_2^{mn} = \frac{2\pi}{R^2}\frac{1}{|k|^2}\eta^{mn}, \tag{2.150}$$

and another from the ghost interactions

$$G_{2A}{}^{B} = -\frac{i}{\bar{k}}\left(F_A{}^C G_{1C}{}^B\right) = \frac{2\pi}{R^2}\frac{1}{\bar{k}^2}\hat{N}_A^B = G_2^B{}_A, \tag{2.151}$$
$$G_{2A\hat{A}} = -\frac{i}{\bar{k}}\left(F_{A\hat{C}}G_1^{\hat{C}}{}_{\hat{A}}\right) = -\frac{2\pi}{R^2}\frac{1}{|k|^2}\omega_B\hat{\omega}_{\hat{B}}A_{A\hat{A}}^{B\hat{B}} = G_{2\hat{A}A}, \tag{2.152}$$
$$G_{2A}{}^{\hat{B}} = -\frac{i}{\bar{k}}\left(F_A{}^{\hat{C}}G_{1\hat{C}}{}^{\hat{B}}\right) = \frac{2\pi}{R^2}\frac{1}{|k|^2}\omega_B\hat{\lambda}^{\hat{A}}A_{A\hat{A}}^{B\hat{B}} = G_2^{\hat{B}}{}_A, \tag{2.153}$$
$$G_2^B{}_{\hat{A}} = \frac{i}{\bar{k}}\left(F^B_{\hat{C}}G_1^{\hat{C}}{}_{\hat{A}}\right) = \frac{2\pi}{R^2}\frac{1}{|k|^2}\lambda^A\hat{\omega}_{\hat{B}}A_{A\hat{A}}^{B\hat{B}} = G^B_{2\hat{A}}, \tag{2.154}$$
$$G_2^{B\hat{B}} = \frac{i}{\bar{k}}\left(F^{B\hat{C}}G_{1\hat{C}}{}^{\hat{B}}\right) = -\frac{2\pi}{R^2}\frac{1}{|k|^2}\lambda^A\hat{\lambda}^{\hat{A}}A_{A\hat{A}}^{B\hat{B}} = G_2^{\hat{B}B}, \tag{2.155}$$
$$G_{2\hat{A}}{}^{\hat{B}} = -\frac{i}{k}\left(F_{\hat{A}}{}^{\hat{C}}G_{1\hat{C}}{}^{\hat{B}}\right) = \frac{2\pi}{R^2}\frac{1}{k^2}N_{\hat{A}}^{\hat{B}} = G_2^{\hat{B}}{}_{\hat{A}}. \tag{2.156}$$

At order $1/k^3$ we have interaction between the non-ghost fields. We organize these terms in the same order as in the previous section, when $G_{\Lambda\Omega} = cG_{\Omega\Lambda}$, with $c = \pm 1$ we only list the first term.

Using the given prescription, we find that the $G_3^{\alpha\Lambda}$ terms are

$$G_3^{\alpha\beta} = -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{\hat{\beta}\hat{\alpha}}G_2^{\hat{\beta}\beta}\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{\bar{J}_2^m}{\bar{k}}f_{m\hat{\alpha}\hat{\beta}}\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}}, \tag{2.157}$$
$$G_3^{\alpha m} = -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{n\hat{\alpha}}G_2^{nm}\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{\bar{J}_3^{\hat{\beta}}}{\bar{k}}f_{n\hat{\alpha}\hat{\beta}}\eta^{\alpha\hat{\alpha}}\eta^{mn} = -G_3^{m\alpha}, \tag{2.158}$$
$$G_3^{\alpha\hat{\alpha}} = -\frac{\eta^{\alpha\hat{\beta}}}{|k|^2}\left(F_{\beta\hat{\beta}}G_2^{\beta\hat{\alpha}}\right) = \frac{2\pi}{R^2}\frac{i}{|k|^2}\left(\frac{N^i}{k} + \frac{\hat{N}^i}{\bar{k}}\right)f_{i\beta\hat{\beta}}\eta^{\alpha\hat{\beta}}\eta^{\beta\hat{\alpha}} = G_3^{\hat{\alpha}\alpha}, \tag{2.159}$$
$$G_3^{\alpha}{}_A = -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{B\hat{\alpha}}G_1^B{}_A\right) = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{\bar{J}_1^{\beta}}{\bar{k}}\omega_B A_A{}^B{}_{\beta\hat{\alpha}}\eta^{\alpha\hat{\alpha}} = -G_{3A}{}^{\alpha}, \tag{2.160}$$
$$G_3^{\alpha B} = -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F^A_{\hat{\alpha}}G_{1A}{}^B\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{\bar{J}_1^{\beta}}{\bar{k}}\lambda^A A_A{}^B{}_{\beta\hat{\alpha}}\eta^{\alpha\hat{\alpha}} = -G_3^{B\alpha}, \tag{2.161}$$

$$G_3^{\alpha}{}_{\hat{A}} = -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{\hat{B}\hat{\alpha}}G_1^{\hat{B}}{}_{\hat{A}}\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{J_1^{\beta}}{k}\hat{\omega}_{\hat{B}}A_{\hat{A}}{}^{\hat{B}}{}_{\beta\hat{\alpha}}\eta^{\alpha\hat{\alpha}} = -G_{3\hat{A}}{}^{\alpha}, \tag{2.162}$$

$$G_3^{\alpha\hat{B}} = -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F^{\hat{A}}{}_{\hat{\alpha}}G_{1\hat{A}}{}^{\hat{B}}\right) = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{J_1^{\beta}}{k}\hat{\lambda}^{\hat{A}}A_{\hat{A}}{}^{\hat{B}}{}_{\beta\hat{\alpha}}\eta^{\alpha\hat{\alpha}} = -G_3^{\hat{B}\alpha}. \tag{2.163}$$

For the $G_3^{m\Lambda}$ terms we find

$$G_3^{mn} = -\frac{\eta^{mp}}{|k|^2}\left(F_{qp}G_2^{qn}\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\left(\frac{N^i}{k}+\frac{\hat{N}^i}{\bar{k}}\right)f_{ipq}\eta^{mp}\eta^{nq}, \tag{2.164}$$

$$G_3^{m\hat{\alpha}} = -\frac{\eta^{mn}}{|k|^2}\left(F_{\alpha n}G_2^{\alpha\hat{\alpha}}\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{J_1^{\beta}}{k}f_{n\alpha\beta}\eta^{\alpha\hat{\alpha}}\eta^{mn} = -G_3^{\hat{\alpha}m} \tag{2.165}$$

$$G_3^{m}{}_{A} = -\frac{\eta^{mn}}{|k|^2}\left(F_{Bn}G_1^{B}{}_{A}\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{\bar{J}_2^{p}}{\bar{k}}\omega_B A_A{}^{B}{}_{np}\eta^{mn} = -G_{3A}{}^{m}, \tag{2.166}$$

$$G_3^{mB} = -\frac{\eta^{mn}}{|k|^2}\left(F^{A}{}_{\hat{\alpha}}G_{1A}{}^{B}\right) = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{\bar{J}_2^{p}}{\bar{k}}\lambda^A A_A{}^{B}{}_{np}\eta^{mn} = G_3^{Bm}, \tag{2.167}$$

$$G_3^{m}{}_{\hat{A}} = -\frac{\eta^{mn}}{|k|^2}\left(F_{\hat{B}\hat{\alpha}}G_1^{\hat{B}}{}_{\hat{A}}\right) = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{J_2^{p}}{k}\hat{\omega}_{\hat{B}}A_{\hat{A}}{}^{\hat{B}}{}_{np}\eta^{mn} = -G_{3\hat{A}}{}^{m}, \tag{2.168}$$

$$G_3^{m\hat{B}} = -\frac{\eta^{mn}}{|k|^2}\left(F^{\hat{A}}{}_{\hat{\alpha}}G_{1\hat{A}}{}^{\hat{B}}\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{J_2^{p}}{k}\hat{\lambda}^{\hat{A}}A_{\hat{A}}{}^{\hat{B}}{}_{np}\eta^{mn} = -G_3^{\hat{B}m}. \tag{2.169}$$

The $G_3^{\hat{\alpha}\Lambda}$ terms computed are

$$G_3^{\hat{\alpha}\hat{\beta}} = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{J_2^{m}}{k}f_{m\alpha\beta}\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}} \tag{2.170}$$

$$G_3^{\hat{\alpha}}{}_{A} = \frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{B\hat{\alpha}}G_1^{B}{}_{A}\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{\bar{J}_3^{\hat{\beta}}}{\bar{k}}\omega_B A_A{}^{B}{}_{\hat{\beta}\alpha}\eta^{\alpha\hat{\alpha}} = -G_{3A}{}^{\hat{\alpha}}, \tag{2.171}$$

$$G_3^{\hat{\alpha}B} = \frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F^{A}{}_{\hat{\alpha}}G_{1A}{}^{B}\right) = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{\bar{J}_3^{\hat{\beta}}}{\bar{k}}\lambda^A A_A{}^{B}{}_{\hat{\beta}\alpha}\eta^{\alpha\hat{\alpha}} = -G_3^{B\hat{\alpha}}, \tag{2.172}$$

$$G_3^{\hat{\alpha}}{}_{\hat{A}} = \frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{\hat{B}\hat{\alpha}}G_1^{\hat{B}}{}_{\hat{A}}\right) = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{J_3^{\hat{\beta}}}{k}\hat{\omega}_{\hat{B}}A_{\hat{A}}{}^{\hat{B}}{}_{\hat{\beta}\alpha}\eta^{\alpha\hat{\alpha}} = -G_{3\hat{A}}{}^{\hat{\alpha}}, \tag{2.173}$$

$$G_3^{\hat{\alpha}\hat{B}} = \frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F^{\hat{A}}{}_{\hat{\alpha}}G_{1\hat{A}}{}^{\hat{B}}\right) = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{J_3^{\hat{\beta}}}{k}\hat{\lambda}^{\hat{A}}A_{\hat{A}}{}^{\hat{B}}{}_{\hat{\beta}\alpha}\eta^{\alpha\hat{\alpha}} = -G_3^{\hat{B}\hat{\alpha}}, \tag{2.174}$$

$$\tag{2.175}$$

The G_3 with only ghost indices are

$$G_{3AC} = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_B\omega_D\hat{\lambda}^{\hat{A}}\hat{\omega}_{\hat{B}}\left[A^{B\hat{C}}_{A\hat{A}}A^{D\hat{B}}_{C\hat{C}} - A^{B\hat{B}}_{A\hat{C}}A^{D\hat{C}}_{C\hat{A}}\right], \tag{2.176}$$

$${G_{3A}}^{B} = \frac{2\pi}{R^2}\frac{i}{\bar{k}^3}\left(\delta^D_A\bar{\partial} - \hat{N}^D_A\right)\hat{N}^B_D + \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_D\lambda^C\hat{\lambda}^{\hat{A}}\hat{\omega}_{\hat{B}}\left[A^{D\hat{B}}_{A\hat{C}}A^{B\hat{C}}_{C\hat{A}} - A^{D\hat{C}}_{A\hat{A}}A^{B\hat{B}}_{C\hat{C}}\right], \tag{2.177}$$

$$G_{3A\hat{A}} = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\left(\delta^D_A\bar{\partial} - \hat{N}^D_A\right)\omega_B\hat{\omega}_{\hat{B}}A^{B\hat{B}}_{D\hat{A}} - \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\omega_B\hat{\omega}_{\hat{B}}N^{\hat{D}}_{\hat{A}}A^{B\hat{B}}_{A\hat{D}}, \tag{2.178}$$

$${G_{3A}}^{\hat{B}} = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\left(\delta^D_A\bar{\partial} - \hat{N}^D_A\right)\omega_B\hat{\lambda}^{\hat{A}}A^{B\hat{B}}_{D\hat{A}} - \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\omega_B\hat{\lambda}^{\hat{A}}N^{\hat{B}}_{\hat{D}}A^{B\hat{D}}_{A\hat{A}}, \tag{2.179}$$

$${G^B_3}_A = \frac{2\pi}{R^2}\frac{i}{\bar{k}^3}\left(\delta^B_D\bar{\partial} + \hat{N}^B_D\right)\hat{N}^D_A + \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_D\lambda^C\hat{\lambda}^{\hat{A}}\hat{\omega}_{\hat{B}}\left[A^{D\hat{C}}_{A\hat{A}}A^{B\hat{B}}_{C\hat{C}} - A^{D\hat{B}}_{A\hat{C}}A^{B\hat{C}}_{C\hat{A}}\right], \tag{2.180}$$

$$G^{BD}_3 = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\lambda^A\lambda^C\hat{\lambda}^{\hat{A}}\hat{\omega}_{\hat{B}}\left[A^{B\hat{C}}_{A\hat{A}}A^{D\hat{B}}_{C\hat{C}} - A^{B\hat{B}}_{A\hat{C}}A^{D\hat{C}}_{C\hat{A}}\right], \tag{2.181}$$

$${G^B_3}_{\hat{A}} = \frac{2\pi}{R^2}\frac{i}{\bar{k}}\frac{1}{|k|^2}\left(\delta^B_D\bar{\partial} + \hat{N}^B_D\right)\lambda^A\hat{\omega}_{\hat{B}}A^{B\hat{B}}_{A\hat{A}} + \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\lambda^A\hat{\omega}_{\hat{B}}N^{\hat{D}}_{\hat{A}}A^{B\hat{B}}_{A\hat{D}}, \tag{2.182}$$

$$G^{B\hat{B}}_3 = -\frac{2\pi}{R^2}\frac{i}{\bar{k}}\frac{1}{|k|^2}\left(\delta^B_D\bar{\partial} + \hat{N}^B_D\right)\lambda^A\hat{\lambda}^{\hat{A}}A^{B\hat{B}}_{A\hat{A}} + \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\lambda^A\hat{\lambda}^{\hat{A}}N^{\hat{B}}_{\hat{D}}A^{B\hat{D}}_{A\hat{A}}, \tag{2.183}$$

$$G_{3\hat{A}A} = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\left(-\delta^{\hat{D}}_{\hat{A}}\partial + N^{\hat{D}}_{\hat{A}}\right)\omega_B\hat{\omega}_{\hat{B}}A^{B\hat{B}}_{A\hat{A}} - \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_B\hat{\omega}_{\hat{B}}\hat{N}^D_A A^{B\hat{B}}_{D\hat{A}}, \tag{2.184}$$

$${G_{3\hat{A}}}^{B} = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\left(-\delta^{\hat{D}}_{\hat{A}}\partial + N^{\hat{D}}_{\hat{A}}\right)\lambda^A\hat{\omega}_{\hat{B}}A^{B\hat{B}}_{A\hat{A}} - \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\lambda^A\hat{\omega}_{\hat{B}}\hat{N}^B_C A^{C\hat{B}}_{A\hat{A}}, \tag{2.185}$$

$$G_{3\hat{A}\hat{C}} = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\hat{\omega}_{\hat{B}}\hat{\omega}_{\hat{D}}\lambda^A\omega_B\left[A^{C\hat{B}}_{A\hat{A}}A^{B\hat{D}}_{C\hat{C}} - A^{B\hat{B}}_{C\hat{A}}A^{C\hat{D}}_{A\hat{C}}\right], \tag{2.186}$$

$${G_{3\hat{A}}}^{\hat{B}} = -\frac{2\pi}{R^2}\frac{i}{k^3}\left(-\delta^{\hat{D}}_{\hat{A}}\partial + N^{\hat{D}}_{\hat{A}}\right)N^{\hat{B}}_{\hat{D}} + \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\hat{\omega}_{\hat{D}}\hat{\lambda}^{\hat{C}}\lambda^A\omega_B\left[A^{B\hat{D}}_{C\hat{A}}A^{C\hat{B}}_{A\hat{C}} - A^{C\hat{D}}_{A\hat{A}}A^{B\hat{B}}_{C\hat{C}}\right], \tag{2.187}$$

$${G^{\hat{B}}_3}_A = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\left(\delta^{\hat{B}}_{\hat{D}}\partial + N^{\hat{B}}_{\hat{D}}\right)\omega_B\hat{\lambda}^{\hat{A}}A^{B\hat{D}}_{A\hat{A}} + \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_B\hat{\lambda}^{\hat{A}}\hat{N}^C_A A^{B\hat{B}}_{C\hat{A}}, \tag{2.188}$$

$$G^{\hat{B}B}_3 = -\frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\left(\delta^{\hat{B}}_{\hat{D}}\partial + N^{\hat{B}}_{\hat{D}}\right)\lambda^A\hat{\lambda}^{\hat{A}}A^{B\hat{D}}_{A\hat{A}} + \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{\bar{k}}\lambda^A\hat{\lambda}^{\hat{A}}\hat{N}^B_A A^{C\hat{B}}_{A\hat{A}}, \tag{2.189}$$

$${G^{\hat{B}}_3}_{\hat{A}} = \frac{2\pi}{R^2}\frac{i}{k^3}\left(\delta^{\hat{B}}_{\hat{D}}\partial + N^{\hat{B}}_{\hat{D}}\right)N^{\hat{D}}_{\hat{A}} - \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\hat{\omega}_{\hat{D}}\hat{\lambda}^{\hat{C}}\lambda^A\omega_B\left[A^{C\hat{B}}_{A\hat{C}}A^{B\hat{D}}_{C\hat{A}} - A^{B\hat{B}}_{C\hat{C}}A^{C\hat{B}}_{A\hat{A}}\right], \tag{2.190}$$

$$G^{\hat{B}\hat{D}}_3 = \frac{2\pi}{R^2}\frac{i}{|k|^2}\frac{1}{k}\hat{\lambda}^{\hat{A}}\hat{\lambda}^{\hat{C}}\lambda^A\omega_B\left[A^{C\hat{B}}_{A\hat{A}}A^{B\hat{D}}_{C\hat{C}} - A^{B\hat{B}}_{C\hat{A}}A^{C\hat{D}}_{A\hat{C}}\right]. \tag{2.191}$$

The $1/k^4$ terms are needed when we compute terms with two derivatives. Since we are not computing anything with two derivatives and at least one ghost field, we don't list those Green's functions. The $G_4^{\alpha\Lambda}$ terms are:

$$
\begin{aligned}
G_4^{\alpha\beta} =& \frac{2\pi}{R^2}\frac{1}{|k|^2\bar{k}^2}\left(\bar{\partial}\bar{J}_2^m f_{m\hat{\alpha}\hat{\beta}} + \bar{J}_2^m\hat{N}^i\left[f_{i\mu\hat{\alpha}}f_{m\hat{\mu}\hat{\beta}} - f_{i\mu\hat{\beta}}f_{m\hat{\mu}\hat{\alpha}}\right]\eta^{\mu\hat{\mu}} + \bar{J}_3^{\hat{\mu}}\bar{J}_3^{\hat{\nu}}f_{m\hat{\alpha}\hat{\mu}}f_{n\hat{\beta}\hat{\nu}}\eta^{mn}\right)\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}} \\
&+ \frac{2\pi}{R^2}\frac{1}{|k|^4}\left(\frac{1}{2}\partial\bar{J}_2^m f_{m\hat{\alpha}\hat{\beta}} + \frac{1}{2}J_1^\mu\bar{J}_1^\nu g^{ij}\left(f_{i\mu\hat{\alpha}}f_{j\nu\hat{\beta}} - f_{i\nu\hat{\alpha}}f_{j\mu\hat{\beta}}\right)\right. \\
&\left.+\frac{1}{2}\left[-\bar{J}_2^m N^i + J_2^m\hat{N}^i\right]\left(f_{m\hat{\alpha}\hat{\mu}}f_{i\mu\hat{\beta}} - f_{m\hat{\beta}\hat{\mu}}f_{i\mu\hat{\alpha}}\right)\eta^{\mu\hat{\mu}}\right)\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}},
\end{aligned} \tag{2.192}
$$

$$
\begin{aligned}
G_4^{\alpha m} =& \frac{2\pi}{R^2}\frac{1}{|k|^2\bar{k}^2}\left[\bar{\partial}\bar{J}_3^{\hat{\beta}}f_{n\hat{\alpha}\hat{\beta}} + \bar{J}_3^{\hat{\beta}}\hat{N}^i\left(f_{p\hat{\alpha}\hat{\beta}}f_{inq}\eta^{pq} + f_{n\hat{\mu}\hat{\beta}}f_{i\mu\hat{\alpha}}\eta^{\mu\hat{\mu}}\right)\right]\eta^{mn}\eta^{\alpha\hat{\alpha}} \\
&+ \frac{2\pi}{R^2}\frac{1}{|k|^4}\left[\frac{1}{8}\left(3\partial\bar{J}_3^{\hat{\beta}} + \bar{\partial}J_3^{\hat{\beta}}\right)f_{n\hat{\alpha}\hat{\beta}} - \frac{1}{2}(3N^i\bar{J}_3^{\hat{\beta}} + \hat{N}^iJ_3^{\hat{\beta}})\left(f_{ipn}f_{q\hat{\alpha}\hat{\beta}}\eta^{pq} - f_{i\hat{\alpha}\mu}f_{n\hat{\beta}\hat{\mu}}\eta^{\mu\hat{\mu}}\right)\right. \\
&\left.-\frac{1}{8}\left[5\bar{J}_1^\beta J_2^p + 3J_1^\beta\bar{J}_2^p\right]f_{i\beta\hat{\alpha}}f_{jnp}g^{ij} - \frac{1}{8}\left[3\bar{J}_1^\beta J_2^p + 5J_1^\beta\bar{J}_2^p\right]f_{n\beta\mu}f_{p\hat{\alpha}\hat{\mu}}\eta^{\mu\hat{\mu}}\right]\eta^{\alpha\hat{\alpha}}\eta^{mn},
\end{aligned} \tag{2.193}
$$

$$
\begin{aligned}
G_4^{\alpha\hat{\alpha}} =& \frac{2\pi}{R^2}\frac{1}{|k|^2k^2}\left[-\partial N^i f_{i\beta\hat{\beta}} - N^iN^j f_{i\mu\hat{\beta}}f_{j\beta\hat{\mu}}\eta^{\mu\hat{\mu}}\right]\eta^{\alpha\hat{\beta}}\eta^{\beta\hat{\alpha}} \\
&+ \frac{2\pi}{R^2}\frac{1}{|k|^2\bar{k}^2}\left[-\bar{\partial}\hat{N}^i f_{i\beta\hat{\beta}} - \hat{N}^i\hat{N}^j f_{i\mu\hat{\beta}}f_{j\beta\hat{\mu}}\eta^{\mu\hat{\mu}}\right]\eta^{\alpha\hat{\beta}}\eta^{\beta\hat{\alpha}} \\
&+ \frac{2\pi}{R^2}\frac{1}{|k|^4}\left[-\left(N^i\hat{N}^j + N^j\hat{N}^i\right)f_{i\mu\hat{\beta}}f_{j\hat{\mu}\beta}\eta^{\mu\hat{\mu}} + \frac{1}{4}J_2^m\bar{J}_2^n\left(3f_{m\mu\beta}f_{n\hat{\beta}\hat{\mu}} + f_{n\mu\beta}f_{m\hat{\beta}\hat{\mu}}\right)\eta^{\mu\hat{\mu}}\right. \\
&\left.+\frac{1}{4}J_1^\mu\bar{J}_3^{\hat{\mu}}\left(5f_{m\mu\beta}f_{n\hat{\mu}\hat{\beta}}\eta^{mn} - f_{i\mu\hat{\beta}}f_{j\hat{\mu}\beta}g^{ij}\right) - \frac{1}{4}\bar{J}_1^\mu J_3^{\hat{\mu}}\left(f_{m\mu\beta}f_{n\hat{\mu}\hat{\beta}}\eta^{mn} + 3f_{i\mu\hat{\beta}}f_{j\hat{\mu}\beta}g^{ij}\right)\right]\eta^{\alpha\hat{\beta}}\eta^{\beta\hat{\alpha}}.
\end{aligned} \tag{2.194}
$$

The $G_4^{m\Lambda}$ Green's functions are

$$
\begin{aligned}
G_4^{mn} =& \frac{2\pi}{R^2}\frac{1}{|k|^2\bar{k}^2}\left[\bar{\partial}\hat{N}^i f_{ipq} - \hat{N}^i\hat{N}^j f_{irp}f_{jsq}\eta^{rs}\right]\eta^{nq}\eta^{mp} \\
&+ \frac{2\pi}{R^2}\frac{1}{|k|^2k^2}\left[\partial N^i f_{ipq} - N^iN^j f_{irp}f_{jsq}\eta^{rs}\right]\eta^{nq}\eta^{mp} \\
&+ \frac{2\pi}{R^2}\frac{1}{|k|^4}\eta^{mp}\eta^{nq}\left[-\left(N^i\hat{N}^j + N^j\hat{N}^i\right)f_{irp}f_{jsq}\eta^{rs} + \frac{1}{2}J_2^r\bar{J}_2^s\left(f_{irp}f_{jsq} + f_{irq}f_{jsp}\right)g^{ij}\right. \\
&\left.-\frac{1}{2}\left(J_1^\alpha\bar{J}_3^{\hat{\alpha}} + \bar{J}_1^\alpha J_3^{\hat{\alpha}}\right)\left(f_{q\alpha\beta}f_{p\hat{\alpha}\hat{\beta}} + f_{p\alpha\beta}f_{q\hat{\alpha}\hat{\beta}}\right)\eta^{\beta\hat{\beta}}\right],
\end{aligned} \tag{2.195}
$$

$$
G_4^{\hat{\alpha}m} = \frac{2\pi}{R^2}\frac{1}{|k|^2k^2}\left[-\partial J_1^\beta f_{n\alpha\beta} + J_1^\beta N^i\left(f_{ipn}f_{q\alpha\beta}\eta^{pq} + f_{n\mu\beta}f_{i\hat{\mu}\alpha}\eta^{\mu\hat{\mu}}\right)\right]\eta^{mn}\eta^{\alpha\hat{\alpha}}
$$

$$
\begin{aligned}
&+\frac{2\pi}{R^2}\frac{1}{|k|^4}\left[-\frac{1}{8}\left(3\bar{\partial}J_1^{\beta}+\partial\bar{J}_1^{\beta}\right)f_{n\alpha\beta}+\frac{1}{2}\left(N^i\bar{J}_1^{\beta}+3\hat{N}^iJ_1^{\beta}\right)\left(f_{ipn}f_{q\alpha\beta}\eta^{pq}+f_{i\alpha\hat{\mu}}f_{n\beta\mu}\eta^{\mu\hat{\mu}}\right)\right.\\
&\left.+\frac{1}{8}\left(3J_2^p\bar{J}_3^{\hat{\beta}}+5\bar{J}_2^pJ_3^{\hat{\beta}}\right)f_{i\alpha\hat{\beta}}f_{jnp}g^{ij}-\frac{1}{8}\left(5J_2^p\bar{J}_3^{\hat{\beta}}+3\bar{J}_2^pJ_3^{\hat{\beta}}\right)f_{p\alpha\mu}f_{n\hat{\beta}\hat{\mu}}\eta^{\mu\hat{\mu}}\right]\eta^{\alpha\hat{\alpha}}\eta^{nm}.
\end{aligned}
\tag{2.196}
$$

Finally, we list the $G_4^{\hat{\alpha}\hat{\beta}}$ term

$$
\begin{aligned}
G_4^{\hat{\alpha}\hat{\beta}}=&\frac{2\pi}{R^2}\frac{1}{|k|^2k^2}\left[\partial J_2^m f_{m\alpha\beta}+J_2^mN^i\left(f_{m\alpha\mu}f_{i\beta\hat{\mu}}-f_{m\beta\mu}f_{i\alpha\hat{\mu}}\right)\eta^{\mu\hat{\mu}}+J_1^{\mu}J_1^{\nu}f_{m\alpha\mu}f_{n\beta\nu}\eta^{mn}\right]\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}}\\
&+\frac{2\pi}{R^2}\frac{1}{|k|^4}\left[\frac{1}{2}\bar{\partial}J_2^m f_{m\alpha\beta}+\frac{1}{2}J_3^{\hat{\mu}}\bar{J}_3^{\hat{\nu}}\left(f_{i\hat{\mu}\alpha}f_{j\hat{\nu}\beta}-f_{i\hat{\mu}\beta}f_{j\hat{\nu}\alpha}g^{ij}\right)\right.\\
&\left.+\frac{1}{2}\left(N^i\bar{J}_2^m-\hat{N}^iJ_2^m\right)\left(f_{m\beta\mu}f_{i\alpha\hat{\mu}}-f_{m\alpha\mu}f_{i\beta\hat{\mu}}\right)\eta^{\mu\hat{\mu}}\right]\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}}.
\end{aligned}
\tag{2.197}
$$

The reason we don't compute terms such as $G_4^{\hat{\alpha}m}$ is that we can deduce their contribution from the relation $\langle\partial X\partial X\rangle=\partial\langle X\partial X\rangle-\langle X\partial\partial X\rangle$, as explained in section 2.

2.C.4 Pairing rules

We split the current in its gauge part J_0 and the vielbein K:

$$J=J_0+K, \tag{2.198}$$

$$K=J_1+J_2+J_3. \tag{2.199}$$

We also join the quantum fluctuations into a single term

$$X=x_1+x_2+x_3. \tag{2.200}$$

The following is the list of all divergent parts up to two derivatives. The order of the results is: first terms with no derivatives, then the currents, then one X with one current, and finally two currents. Finally, we list the pairing rules involving ghost fields. The definition of I in this appendix is $\mathrm{I}=-1/(2R^2\epsilon)$.

The non-vanishing terms with no derivatives are the ones given by the first term in the Schwinger-Dyson equation:

$$\langle x_1, x_3\rangle = -T_\alpha T_{\hat\alpha}\eta^{\alpha\hat\alpha} \qquad \text{and} \qquad \langle x_2, x_2\rangle = T_m T_n \eta^{mn}. \tag{2.201}$$

Now we show the divergent part of the currents:

$$\langle K\rangle = \langle \bar K\rangle = \langle N\rangle = \langle \hat N\rangle = 0, \tag{2.202}$$

$$\langle J_0\rangle = -\frac{\mathrm{I}}{2}\left(\{[N, T_{\hat\alpha}], T_\alpha\}\eta^{\alpha\hat\alpha} - \{[N, T_\alpha], T_{\hat\alpha}\}\eta^{\alpha\hat\alpha} + [[N, T_m], T_n]\eta^{mn}\right), \tag{2.203}$$

$$\langle \bar J_0\rangle = -\frac{\mathrm{I}}{2}\left(\{[\hat N, T_{\hat\alpha}], T_\alpha\}\eta^{\alpha\hat\alpha} - \{[\hat N, T_\alpha], T_{\hat\alpha}\}\eta^{\alpha\hat\alpha} + [[\hat N, T_m], T_n]\eta^{mn}\right). \tag{2.204}$$

For one X with one current, we find that the simplest current is J_0

$$\langle X, J_0\rangle = -\mathrm{I}[K, T_j]T_k g^{jk}, \tag{2.205}$$

$$\langle X, \bar J_0\rangle = -\mathrm{I}[\bar K, T_j]T_k g^{jk}, \tag{2.206}$$

for the other currents we find

$$\langle x_1, J_1\rangle = -\mathrm{I}[J_2, T_{\hat\alpha}]T_\alpha\eta^{\alpha\hat\alpha}, \quad \langle x_2, J_1\rangle = \mathrm{I}[J_3, T_{\hat\alpha}]T_\alpha\eta^{\alpha\hat\alpha}, \quad \langle x_3, J_1\rangle = \mathrm{I}[N, T_{\hat\alpha}]T_\alpha\eta^{\alpha\hat\alpha}, \tag{2.207}$$

$$\langle x_1, \bar J_1\rangle = 0, \quad \langle x_2, \bar J_1\rangle = 0, \quad \langle x_3, \bar J_1\rangle = \mathrm{I}[\hat N, T_{\hat\alpha}]T_\alpha\eta^{\alpha\hat\alpha}, \tag{2.208}$$

$$\langle x_1, J_2\rangle = -\mathrm{I}[J_3, T_m]T_n\eta^{mn}, \quad \langle x_2, J_2\rangle = \mathrm{I}[N, T_m]T_n\eta^{mn}, \quad \langle x_3, J_2\rangle = 0, \tag{2.209}$$

$$\langle x_1, \bar J_2\rangle = 0, \quad \langle x_2, \bar J_2\rangle = \mathrm{I}[\hat N, T_m]T_n\eta^{mn}, \quad \langle x_3, \bar J_2\rangle = \mathrm{I}[\bar J_1, T_m]T_n\eta^{mn}, \tag{2.210}$$

$$\langle x_1, J_3\rangle = -\mathrm{I}[N, T_\alpha]T_{\hat\alpha}\eta^{\alpha\hat\alpha}, \quad \langle x_2, J_3\rangle = 0, \quad \langle x_3, J_3\rangle = 0, \tag{2.211}$$

$$\langle x_1, \bar J_3\rangle = -\mathrm{I}[\hat N, T_\alpha]T_{\hat\alpha}\eta^{\alpha\hat\alpha}, \quad \langle x_2, \bar J_3\rangle = \mathrm{I}[\bar J_1, T_\alpha]T_{\hat\alpha}\eta^{\alpha\hat\alpha}, \quad \langle x_3, \bar J_3\rangle = \mathrm{I}[\bar J_2, T_\alpha]T_{\hat\alpha}\eta^{\alpha\hat\alpha}. \tag{2.212}$$

Now we show the divergent part of two currents. The first group are the $\langle J_0, \cdot\rangle$ terms:

$$\langle J_0, J_0\rangle = \mathrm{I}[J_1, T_{\hat\alpha}][J_3, T_\alpha]\eta^{\alpha\hat\alpha} - \mathrm{I}[J_3, T_\alpha][J_1, T_{\hat\alpha}]\eta^{\alpha\hat\alpha} + \mathrm{I}[J_2, T_m][J_2, T_n]\eta^{mn}, \tag{2.213}$$

$$\langle J_0, J_1\rangle = -\mathrm{I}[J_1, T_{\hat\alpha}][N, T_\alpha]\eta^{\alpha\hat\alpha} - \mathrm{I}[J_3, T_\alpha][J_2, T_{\hat\alpha}]\eta^{\alpha\hat\alpha} + \mathrm{I}[J_2, T_m][J_3, T_n]\eta^{mn}, \tag{2.214}$$

$$\langle J_0, \bar J_1\rangle = -\mathrm{I}[J_1, T_{\hat\alpha}][\hat N, T_\alpha]\eta^{\alpha\hat\alpha}, \tag{2.215}$$

$$\langle J_0, J_2\rangle = -\mathrm{I}[J_3, T_\alpha][J_3, T_{\hat\alpha}]\eta^{\alpha\hat\alpha} - \mathrm{I}[J_2, T_m][N, T_n]\eta^{mn}, \tag{2.216}$$

$$\langle J_0, \bar J_2\rangle = \mathrm{I}[J_1, T_{\hat\alpha}][\bar J_1, T_\alpha]\eta^{\alpha\hat\alpha} - \mathrm{I}[J_2, T_m][\hat N, T_n]\eta^{mn}, \tag{2.217}$$

$$\langle J_0, J_3\rangle = \mathrm{I}[J_3, T_\alpha][N, T_{\hat\alpha}]\eta^{\alpha\hat\alpha}, \tag{2.218}$$

$$\langle J_0, \bar J_3\rangle = \mathrm{I}[J_3, T_\alpha][\hat N, T_{\hat\alpha}]\eta^{\alpha\hat\alpha} + \mathrm{I}[J_1, T_{\hat\alpha}][\bar J_2, T_\alpha]\eta^{\alpha\hat\alpha} + \mathrm{I}[J_2, T_m][\bar J_1, T_n]\eta^{mn}. \tag{2.219}$$

The $\langle J_1, \cdot\rangle$ terms are

$$\langle J_1, J_1\rangle = -\mathrm{I}\left([J_2, T_{\hat\alpha}][N, T_\alpha] - [N, T_\alpha][J_2, T_{\hat\alpha}]\right)\eta^{\alpha\hat\alpha} + [J_3, T_m][J_3, T_n]\eta^{mn}, \tag{2.220}$$

$$\langle \bar J_1, \bar J_1\rangle = 0, \tag{2.221}$$

$$\begin{aligned}\langle J_1, \bar J_1\rangle =& \frac{\mathrm{I}}{2}[\partial J_2, T_{\hat\alpha}]T_\alpha\eta^{\alpha\hat\alpha} + \frac{\mathrm{I}}{2}\left([J_1, T_i][\bar J_1, T_j] + [\bar J_1, T_i][J_1, T_j]\right)g^{ij}\\ &+ \frac{\mathrm{I}}{2}\Big(-[\bar J_2, T_{\hat\alpha}][N, T_\alpha] - [J_2, T\hat\alpha][\hat N, T_\alpha] + 3[N, T_\alpha][\bar J_2, T_{\hat\alpha}] - [\hat N, T_\alpha][J_2, T_{\hat\alpha}]\Big)\eta^{\alpha\hat\alpha},\end{aligned} \tag{2.222}$$

$$\begin{aligned}\langle \bar J_1, J_1\rangle =& -\frac{\mathrm{I}}{2}[\partial J_2, T_{\hat\alpha}]T_\alpha\eta^{\alpha\hat\alpha} + \frac{\mathrm{I}}{2}\left([J_1, T_i][\bar J_1, T_j] + [\bar J_1, T_i][J_1, T_j]\right)g^{ij}\\ &+ \frac{\mathrm{I}}{2}\Big(-[\bar J_2, T_{\hat\alpha}][N, T_\alpha] - [J_2, T_{\hat\alpha}][\hat N, T_\alpha] + 3[\hat N, T_\alpha][J_2, T_\alpha] - [N, T_\alpha][\bar J_2, T_{\hat\alpha}]\Big)\eta^{\alpha\hat\alpha},\end{aligned} \tag{2.223}$$

$$\langle J_1, J_2\rangle = \mathrm{I}[N, T_\alpha][J_2, T_{\hat\alpha}]\eta^{\alpha\hat\alpha} + [J_3, T_m][J_3, T_n]\eta^{mn}, \tag{2.224}$$

$$\langle \bar J_1, \bar J_2\rangle = 0, \tag{2.225}$$

$$\begin{aligned}\langle J_1, \bar J_2\rangle =& \frac{\mathrm{I}}{8}[5\partial\bar J_3 - \bar\partial J_3, T_m]T_n\eta^{mn} + \frac{\mathrm{I}}{8}\left(11[J_2, T_{\hat\alpha}][\bar J_1, T_\alpha] + 5[\bar J_2, T_{\hat\alpha}][J_1, T_\alpha]\right)\eta^{\alpha\hat\alpha}\\ &+ \frac{\mathrm{I}}{8}\left(5[\bar J_1, T_i][J_2, T_j] + 3[J_1, T_i][\bar J_2, T_j]\right)g^{ij} - \frac{\mathrm{I}}{2}\Big([N, T_a][\bar J_3, T_{\hat\alpha}] + [\hat N, T_\alpha][J_3, T_{\hat\alpha}]\Big)\eta^{\alpha\hat\alpha}\\ &+ \frac{\mathrm{I}}{2}\Big(3[\bar J_3, T_m][N, T_n] - [J_3, T_m][\hat N, T_n]\Big)\eta^{mn},\end{aligned} \tag{2.226}$$

$$\begin{aligned}\langle \bar J_1, J_2\rangle =& -\frac{\mathrm{I}}{8}[3\partial\bar J_3 + \bar\partial J_3, T_m]T_n\eta^{mn} + \frac{3\mathrm{I}}{8}\left([J_2, T_{\hat\alpha}][\bar J_1, T_\alpha] - [\bar J_2, T_{\hat\alpha}][J_1, T_\alpha]\right)\eta^{\alpha\hat\alpha}\\ &+ \frac{\mathrm{I}}{8}\left(5[\bar J_1, T_i][J_2, T_j] + 3[J_1, T_i][\bar J_2, T_j]\right)g^{ij} - \frac{\mathrm{I}}{2}\Big(3[N, T_a][\bar J_3, T_{\hat\alpha}] - [\hat N, T_\alpha][J_3, T_{\hat\alpha}]\Big)\eta^{\alpha\hat\alpha}\\ &+ \frac{\mathrm{I}}{2}\Big([\bar J_3, T_m][N, T_n] + [J_3, T_m][\hat N, T_n]\Big)\eta^{mn},\end{aligned} \tag{2.227}$$

$$\langle J_1, J_3\rangle = -\mathrm{I}[N, T_\alpha][N, T_{\hat\alpha}]\eta^{\alpha\hat\alpha}, \tag{2.228}$$

$$\langle \bar{J}_1, \bar{J}_3\rangle = -\,\mathrm{I}[\hat{N}, T_\alpha][\hat{N}, T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}}, \tag{2.229}$$

$$\begin{aligned}\langle J_1, \bar{J}_3\rangle = &-\mathrm{I}\left([N, T_\alpha][\hat{N}, T_{\hat{\alpha}}] + [\hat{N}, T_\alpha][N, T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}} + \frac{\mathrm{I}}{4}\left(3[\bar{J}_2, T\hat{\alpha}][J_2, T_a] + 5[J_2, T_{\hat{\alpha}}][\bar{J}_2, T_\alpha]\right)\eta^{\alpha\hat{\alpha}}\\ &+ \frac{\mathrm{I}}{4}\left(5[\bar{J}_3, T_m][J_1, T_n] + 3[J_3, T_m][J_1, T_n]\right)\eta^{mn} + \frac{\mathrm{I}}{4}\left([J_1, T_i][\bar{J}_3, T_j] + 3[\bar{J}_1, T_i][J_3, T_j]\right)g^{ij},\end{aligned} \tag{2.230}$$

$$\begin{aligned}\langle \bar{J}_1, J_3\rangle = &-\mathrm{I}\left([N, T_\alpha][\hat{N}, T_{\hat{\alpha}}] + [\hat{N}, T_\alpha][N, T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}} - \frac{\mathrm{I}}{4}\left([\bar{J}_2, T\hat{\alpha}][J_2, T_a] - [J_2, T_{\hat{\alpha}}][\bar{J}_2, T_\alpha]\right)\eta^{\alpha\hat{\alpha}}\\ &+ \frac{\mathrm{I}}{4}\left([\bar{J}_3, T_m][J_1, T_n] - [J_3, T_m][J_1, T_n]\right)\eta^{mn} + \frac{\mathrm{I}}{4}\left([J_1, T_i][\bar{J}_3, T_j] + 3[\bar{J}_1, T_i][J_3, T_j]\right)g^{ij}.\end{aligned} \tag{2.231}$$

We present the $\langle J_3, \cdot\rangle$ terms before the $\langle J_2, \cdot\rangle$ due to their similarity with the $\langle J_1, \cdot\rangle$ terms:

$$\langle J_3, J_2\rangle = 0, \tag{2.232}$$

$$\langle \bar{J}_3, \bar{J}_2\rangle = -\,\mathrm{I}[\hat{N}, T_{\hat{\alpha}}][\bar{J}_1, T_\alpha]\eta^{\alpha\hat{\alpha}} - \mathrm{I}[\bar{J}_1, T_m][\hat{N}, T_n]\eta^{mn}, \tag{2.233}$$

$$\begin{aligned}\langle J_3, J_2\rangle = &\frac{\mathrm{I}}{8}[5\bar{\partial}J_1 - \partial\bar{J}_1, T_m]Tn\eta^{mn} - \frac{\mathrm{I}}{2}\left([\bar{J}_1, T_m][N, T_n] - 3[J_1, T_m][\hat{N}, T_n]\right)\eta^{mn}\\ &+ \frac{\mathrm{I}}{2}\left([\hat{N}, T_{\hat{\alpha}}][J_1, T_\alpha] + [N, T_{\hat{\alpha}}][\bar{J}_1, T_\alpha]\right)\eta^{\alpha\hat{\alpha}} + \frac{\mathrm{I}}{8}\left(3[\bar{J}_3, T_i][J_2, T_j] + 5[J_3, T_i][\bar{J}_2, T_j]\right)g^{ij}\\ &- \frac{\mathrm{I}}{8}\left(5[J_2, T_\alpha][\bar{J}_3, T_{\hat{\alpha}}] + 11[\bar{J}_2, T_\alpha][J_3, T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}},\end{aligned} \tag{2.234}$$

$$\begin{aligned}\langle J_3, J_2\rangle = &-\frac{\mathrm{I}}{8}[3\bar{\partial}J_1 + \partial\bar{J}_1, T_m]T_n\eta^{mn} + \frac{\mathrm{I}}{2}\left([\bar{J}_1, T_m][N, T_n] + [J_1, T_m][\hat{N}, T_n]\right)\eta^{mn}\\ &+ \frac{\mathrm{I}}{2}\left(3[\hat{N}, T_{\hat{\alpha}}][J_1, T_\alpha] - [N, T_{\hat{\alpha}}][\bar{J}_1, T_\alpha]\right)\eta^{\alpha\hat{\alpha}} + \frac{\mathrm{I}}{8}\left(3[\bar{J}_3, T_i][J_2, T_j] + 5[J_3, T_i][\bar{J}_2, T_j]\right)g^{ij}\\ &+ \frac{3\mathrm{I}}{8}\left([J_2, T_\alpha][\bar{J}_3, T_{\hat{\alpha}}] - [\bar{J}_2, T_\alpha][J_3, T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}},\end{aligned} \tag{2.235}$$

$$\langle J_3, J_3\rangle = 0, \tag{2.236}$$

$$\langle \bar{J}_3, \bar{J}_3\rangle = \mathrm{I}\left([\bar{J}_2, T_\alpha][\hat{N}, T_{\hat{\alpha}}] - [\hat{N}, T_{\hat{\alpha}}][\bar{J}_2, T_\alpha]\right)\eta^{\alpha\hat{\alpha}} + \mathrm{I}[\bar{J}_1, T_m][\bar{J}_1, T_n]\eta^{mn}, \tag{2.237}$$

$$\begin{aligned}\langle \bar{J}_3, J_3\rangle = &-\frac{\mathrm{I}}{2}[\bar{\partial}J_2, T_\alpha]T_{\hat{\alpha}}\eta^{\alpha\hat{\alpha}} + \frac{\mathrm{I}}{2}\left([J_3, T_i][\bar{J}_3, T_j] + [\bar{J}_3, T_i][J_3, T_j]\right)g^{ij}\\ &+ \frac{\mathrm{I}}{2}\left(-[N, T_{\hat{\alpha}}][\bar{J}_2, T_\alpha] - [\hat{N}, T_{\hat{\alpha}}][J_2, T_\alpha] + 3[\bar{J}_2, T_\alpha][N, T_{\hat{\alpha}}] - [J_2, T_\alpha][\hat{N}, T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}},\end{aligned} \tag{2.238}$$

$$\langle J_3, \bar{J}_3\rangle = \frac{\mathrm{I}}{2}[\bar{\partial}J_2, T_\alpha]T_{\hat{\alpha}}\eta^{\alpha\hat{\alpha}} + \frac{\mathrm{I}}{2}\left([J_3, T_i][\bar{J}_3, T_j] + [\bar{J}_3, T_i][J_3, T_j]\right)g^{ij}$$

$$+\frac{\mathrm{I}}{2}\Big(-3[N,T_{\hat\alpha}][\bar J_2,T_\alpha]+[\hat N,T_{\hat\alpha}][J_2,T_\alpha]+[\bar J_2,T_\alpha][N,T_{\hat\alpha}]+[J_2,T_\alpha][\hat N,T_{\hat\alpha}]\Big)\,\eta^{\alpha\hat\alpha}. \tag{2.239}$$

Finally, the remaining $\langle J_2,\cdot\rangle$ terms:

$$\langle J_2,J_2\rangle = \mathrm{I}[N,T_m][N,T_n]\eta^{mn}, \tag{2.240}$$

$$\langle \bar J_2,\bar J_2\rangle = \mathrm{I}[\hat N,T_m][\hat N,T_n]\eta^{mn}, \tag{2.241}$$

$$\begin{aligned}\langle \bar J_2,J_2\rangle =& -\mathrm{I}\Big([N,T_m][\hat N,T_n]+[\hat N,T_m][N,T_n]\Big)\,\eta^{mn}+\frac{\mathrm{I}}{2}\left([J_2,T_i][\bar J_2,T_j]+[\bar J_2,T_i][J_2,T_j]\right)g^{ij}\\ &-\frac{\mathrm{I}}{2}\left([J_1,T_\alpha][\bar J_3,T_{\hat\alpha}]-3[\bar J_3,T_{\hat\alpha}][J_1,T_\alpha]+3[\bar J_1,T_\alpha][J_3,T_{\hat\alpha}]-[J_3,T_{\hat\alpha}][\bar J_1,T_\alpha]\right)\eta^{\alpha\hat\alpha},\end{aligned} \tag{2.242}$$

$$\begin{aligned}\langle J_2,\bar J_2\rangle =& -\mathrm{I}\Big([N,T_m][\hat N,T_n]+[\hat N,T_m][N,T_n]\Big)\,\eta^{mn}+\frac{\mathrm{I}}{2}\left([J_2,T_i][\bar J_2,T_j]+[\bar J_2,T_i][J_2,T_j]\right)g^{ij}\\ &+\frac{\mathrm{I}}{2}\left([\bar J_3,T_{\hat\alpha}][J_1,T_\alpha]-3[J_1,T_\alpha][\bar J_3,T_{\hat\alpha}]+3[J_1,T_\alpha][\bar J_3,T_{\hat\alpha}]-[\bar J_1,T_\alpha][J_3,T_{\hat\alpha}]\right)\eta^{\alpha\hat\alpha}.\end{aligned} \tag{2.243}$$

The terms involving ghost fields that have vanishing anomalous dimension are

$$\langle X,N\rangle=\langle X,\hat N\rangle=\langle\omega,\lambda\rangle=\langle\hat\omega,\hat\lambda\rangle=0, \tag{2.244}$$

$$\langle\omega,J\rangle=\langle\lambda,J\rangle=\langle\hat\omega,\bar J\rangle=\langle\hat\lambda,\bar J\rangle=0, \tag{2.245}$$

$$\langle\omega,\bar J_0\rangle=\langle\lambda,\bar J_0\rangle=\langle\hat\omega,J_0\rangle=\langle\hat\lambda,J_0\rangle=0, \tag{2.246}$$

$$\langle\omega,N\rangle=\langle\lambda,N\rangle=\langle\hat\omega,\hat N\rangle=\langle\hat\lambda,\hat N\rangle=0, \tag{2.247}$$

$$\langle J,N\rangle=\langle\bar J,\hat N\rangle=0. \tag{2.248}$$

The expressions involving two ghosts and no derivatives are

$$\langle\omega,\hat\lambda\rangle=-\mathrm{I}[\omega,T_i][\hat\lambda,T_j]g^{ij},\qquad \langle\lambda,\hat\omega\rangle=-\mathrm{I}[\lambda,T_i][\hat\omega,T_j]g^{ij}, \tag{2.249}$$

$$\langle\omega,\hat\omega\rangle=-\mathrm{I}[\omega,T_i][\hat\omega,T_j]g^{ij},\qquad \langle\lambda,\hat\lambda\rangle=-\mathrm{I}[\lambda,T_i][\hat\lambda,T_j]g^{ij}. \tag{2.250}$$

For one ghost and one current, including the ghost currents,

$$\langle\omega,\bar{K}\rangle = -\,\mathrm{I}[\omega,T_i][\bar{K},T_j]g^{ij}, \qquad \langle\hat{\omega},K\rangle = -\mathrm{I}[\hat{\omega},T_i][K,T_j]g^{ij}, \tag{2.251}$$

$$\langle\lambda,\bar{K}\rangle = -\,\mathrm{I}[\lambda,T_i][\bar{K},T_j]g^{ij}, \qquad \langle\hat{\lambda},K\rangle = -\mathrm{I}[\hat{\lambda},T_i][K,T_j]g^{ij}, \tag{2.252}$$

$$\langle\omega,\hat{N}\rangle = -\,\mathrm{I}[\omega,T_i][\hat{N},T_j]g^{ij}, \qquad \langle\hat{\omega},N\rangle = -\mathrm{I}[\hat{\omega},T_i][N,T_j]g^{ij}, \tag{2.253}$$

$$\langle\lambda,\hat{N}\rangle = -\,\mathrm{I}[\lambda,T_i][\hat{N},T_j]g^{ij}, \qquad \langle\hat{\lambda},N\rangle = -\mathrm{I}[\hat{\lambda},T_i][N,T_j]g^{ij}. \tag{2.254}$$

Finally, the terms with two currents, with at least one ghost current:

$$\langle\bar{K},N\rangle = -\,\mathrm{I}[\bar{K},T_i][N,T_j]g^{ij}, \tag{2.255}$$

$$\langle K,\hat{N}\rangle = -\,\mathrm{I}[K,T_i][\hat{N},T_j]g^{ij}, \tag{2.256}$$

$$\langle N,\hat{N}\rangle = -\,\mathrm{I}[N,T_i][\hat{N},T_j]g^{ij}. \tag{2.257}$$

Chapter 3

Supertwistor description of the *AdS* pure spinor string

3.1 Introduction

The superstring sigma model on AdS spaces is usually described in terms of the supergroup coset $PSU(2,2|4)/SO(1,4) \times SO(5)$. The classical Green-Schwarz and pure spinor formulations are both well understood in terms of this coset. However for some applications, the usual exponential parametrization of the coset elements becomes cumbersome.

In [46] Roiban and Siegel introduced another parametrization for the $AdS_5 \times S^5$ coset in terms of the supergroup $GL(4|4)$. The usefulness of this new formulation is in the fact that the coordinates can be represented in terms of unconstrained matrices. Furthermore, the coordinates transform in the fundamental representation of the superconformal group, like supertwistors.

Depending on the application intended, different sets of coordinates are more useful than others. In the same way that global AdS coordinates and Poincaré patch are useful for different applications. This also extends to the full superspace, *e.g.* chiral vs. nonchiral. The construction of explicit vertex operators for string states depends heavily on these choices.

Since the beginning of the formalism vertex operators for AdS have been discussed [81]. The first nontrivial example was introduced in [69, 82]. Further developments can be found in [70, 72]. The most complete description in the case of supergravity states was given in [71]. In this work the authors show that the ghost number two cohomology can be written in terms of harmonic superspace and a direct dictionary to the dual CFT single trace operators was obtained. The derivation is very lengthy due to the usual exponential parametrization of the coset elements. As advocated by Siegel [83], those results could be simplified using the $GL(4|4)$ description. This is one of the motivations to adapt the pure spinor formalism for this new coset. In this paper we will describe in detail how to achieve this.

This paper is organized as follows. In Section 2 we describe the coset and its basic properties. In Section 3 the symmetries of AdS are discussed in terms of the new coset. The full pure spinor superstring action is constructed in Section 4. In Section 5 we make a few comments on the construction of the vertex operator related to the β-deformations. In Section 6 we conclude the paper and discuss future lines of investigation.

3.2 The $GL(4|4)/(GL(1)\times Sp(2))^2$ coset

Roiban and Siegel proposed a description of the $AdS_5\times S^5$ sigma model in terms of a coset that can be described by standard matrices [46]. The observation is that the $PSU(2,2|4)$ group is a coset by itself (not caring about reality conditions) $GL(4|4)/(GL(1)\times GL(1))$, where the two $GL(1)$ groups are defined by scalar multiplication in the upper and lower blocks. Note that the super-determinant is invariant under the action of both $GL(1)$'s combined. Up to reality conditions (*i.e.* signature) $AdS_5\times S^5$ can be described by [1]

$$\frac{GL(4|4)}{(GL(1)\times Sp(2))^2}\,. \tag{3.1}$$

[1]In our notation $Sp(n)$ describes $2n\times 2n$ matrices, *e.g.* $Sp(1)\simeq SU(2)$.

Note that $Sp(2) = Spin(5)$ (under Wick rotation, $Sp(1,1) = Spin(1,4)$.) Since we have a model with spinors, it is much more natural to work with groups where the spinors transform in the fundamental representation.

The coset elements are denoted by $Z_M{}^A$ where the local $\Lambda_A{}^B$ $(GL(1)\times Sp(2))^2$ transformations act on the right by simple matrix multiplication. The index M is a global $GL(4|4)$ index. We divide both indices under bosonic and fermionic elements $M = (m, \bar{m})$ and $A = (a, \bar{a})$. The $Sp(2)$ invariant matrices will be denoted by Ω_{ab} and $\bar{\Omega}_{\bar{a}\bar{b}}$. There are analog matrices with indices up, which will be denoted by the same symbol. They all satisfy $\Omega\Omega = -\mathbb{I}$ where $\mathbb{I}$ is the identity matrix with appropriate indices. We will omit explicit indices most of the time, only making them explicit when necessary.

The left-invariant currents (invariant under global transformations) are defined by

$$J_A{}^B = Z_A{}^M d\, Z_M{}^B\,, \tag{3.2}$$

where $Z_A{}^M = (Z_M{}^A)^{-1}$.

A variation of the group element Z around a background Z_0 is given by

$$\delta Z_M{}^A = Z_M{}^B X_B{}^A\,, \tag{3.3}$$

where $X_B{}^A$ is given by

$$X_B{}^A = \begin{pmatrix} X_b{}^a & \Theta_b{}^{\bar{a}} \\ \Theta'_{\bar{b}}{}^a & Y_{\bar{b}}{}^{\bar{a}} \end{pmatrix}. \tag{3.4}$$

For these variations to be in the coset, the matrices X and Y must satisfy

$$X^T = -\Omega\, X\, \Omega, \quad Y^T = -\bar{\Omega}\, Y\, \bar{\Omega}\,. \tag{3.5}$$

Since these conditions do not imply that X and Y are traceless, we further impose

$$\operatorname{Tr} X = \operatorname{Tr} Y = 0\,. \tag{3.6}$$

Doing this, we ensure that we work only with variations that are orthogonal to the gauge group.

We want to relate the elements described with the Roiban-Siegel formulation and the ones in the description using the $PSU(2,2|4)/(SO(5)\times SO(1,4))$ coset for the pure spinors. Our notation is closely related to the one adopted in [28]. By construction, it is not hard to see the equivalence between Z and the element $g \in PSU(2,2|4)/(SO(5)\times SO(1,4))$,

$$g \equiv Z\,. \tag{3.7}$$

In order to establish the equivalence between the content of the current in both formalism, we first identify the gauge content in our matrix formalism. Writing the block components of J as

$$J = \begin{pmatrix} J_X & K_1 \\ K_3 & J_Y \end{pmatrix}\,, \tag{3.8}$$

we split the diagonal elements into three irreducible components using the $Sp(2)$ metric Ω. Define for a matrix ${M_a}^b$ its three irreducible components,

$$\langle M \rangle = \frac{1}{2}\left[M - \Omega M^T \Omega\right] - \frac{1}{4}\mathbb{I}\operatorname{Tr} M\,, \tag{3.9}$$

$$(M) = \frac{1}{2}\left[M + \Omega M^T \Omega\right]\,, \tag{3.10}$$

$$\operatorname{Tr} M\,. \tag{3.11}$$

Usually, for any matrix, one can split it in its antisymmetric, its symmetric traceless and its trace part. Here $\langle M \rangle$ is the Ω-antisymmetric, Ω-traceless part of M, (M) is the Ω-symmetric part of M, and of course TrM is the Ω-trace of M. Using those independent structures, we can separate the element of J (3.2) that are pure gauge. we will define

$$K_X = \langle J_X\rangle\,,\quad A_X = (J_X)\,,\quad a_X = \frac{1}{4}\mathrm{Tr}J_X\,, \tag{3.12}$$
$$K_Y = \langle J_Y\rangle\,,\quad A_Y = (J_Y)\,,\quad a_Y = \frac{1}{4}\mathrm{Tr}J_Y\,. \tag{3.13}$$

A_X and a_X are $Sp(2)$ and $GL(1)$ connections respectively. By definition,

$$J_X = K_X + A_X + \mathbb{I}a_x\,. \tag{3.14}$$

By checking its transformation property, we can now relate the diagonal elements in (3.8) with the gauge part of current in the $\mathfrak{psu}(2,2|4)$ algebra,

$$J_0^i \equiv \begin{pmatrix} A_X + \mathbb{I}a_X & 0 \\ 0 & A_Y + \mathbb{I}a_Y \end{pmatrix}\,. \tag{3.15}$$

The rest of the bosonic components are related as,

$$J_2^m \equiv \begin{pmatrix} K_X & 0 \\ 0 & K_Y \end{pmatrix}\,. \tag{3.16}$$

Before we continue, we have to make clear that the $\langle\cdot\rangle$ and $(\cdot)$ operations need to be treated with care when there is a product of fermionic matrices. Take two fermionic matrices A and B, is easy to see that

$$\mathrm{Tr}\left(\frac{1}{2}\left[AB - \Omega B^T A^T \Omega\right] - \frac{1}{4}\mathbb{I}\mathrm{Tr}AB\right) = -\mathrm{Tr}AB \neq 0\,, \tag{3.17}$$
$$\mathrm{Tr}\left(\frac{1}{2}\left[AB + \Omega B^T A^T \Omega\right]\right) = \mathrm{Tr}AB \neq 0\,. \tag{3.18}$$

The solution to this problem is to add a $(-)$ sign when transposing fermionic matrices. Thus, for a product of two fermionic matrices A and B, our three irreducible components read

$$\langle AB\rangle =\frac{1}{2}\left[AB+\Omega B^TA^T\Omega\right]-\frac{1}{4}\mathbb{I}\mathrm{Tr}AB\,, \tag{3.19}$$

$$(AB) =\frac{1}{2}\left[AB-\Omega B^TA^T\Omega\right] \quad \text{and} \quad \mathrm{Tr}AB\,. \tag{3.20}$$

It is not so obvious how to relate the fermionic part of $PSU(2,2|4)$, J_1^{α} and $J_3^{\hat{\alpha}}$, with the nondiagonal terms in (3.8), K_1 and K_3, because they do not have the right $\mathbb{Z}_4$ charge. The matrices which do have the right $\mathbb{Z}_4$ charge are F_1 and F_3 which define K_1 and K_3 as

$$K_1=\frac{1}{\sqrt{2}}\left(F_1-F_3^*\right)E^{-1/4} \quad \text{and} \quad K_3=\frac{1}{\sqrt{2}}\left(F_1^*+F_3\right)E^{1/4}\,, \tag{3.21}$$

where

$$F_1^*=\bar{\Omega}F_1^T\Omega\,, \qquad F_3^*=\Omega F_3^T\bar{\Omega}\,, \tag{3.22}$$

and $E=\mathrm{Sdet}Z$. Now the identification is

$$J_1^{\alpha}\equiv F_1 \quad J_3^{\hat{\alpha}}\equiv F_3\,. \tag{3.23}$$

Following the same idea, we define the Θs as functions of elements with the right $\mathbb{Z}_4$ charge,

$$\Theta=\frac{1}{\sqrt{2}}\left(\theta_1-\theta_3^*\right)E^{-1/4} \quad \text{and} \quad \Theta'=\frac{1}{\sqrt{2}}\left(\theta_1^*+\theta_3\right)E^{1/4}\,. \tag{3.24}$$

In the same way as we related the components of the currents generated by g and Z, we can relate variations of g and Z by

$$x_2^m\equiv\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\,, \tag{3.25}$$

$$x_1^{\alpha} \equiv \theta_1\,, \tag{3.26}$$

$$x_3^{\hat{\alpha}} \equiv \theta_3\,. \tag{3.27}$$

It is easy to check that there are the correct number of bosonic and fermionic variations.

Finally, we need to define the ghosts fields that are essential for the construction of the BRST operator. We define the right and left ghost, along with their conjugated momenta, as ${\lambda_a}^{\bar{a}}$, ${\omega_{\bar{a}}}^{a}$, ${\bar{\lambda}_{\bar{a}}}{}^{a}$, ${\bar{\omega}_a}{}^{\bar{a}}$. The indices in the ghost terms are such that λ has the same indices as F_1 and $\bar{\lambda}$ the same as F_3. The crucial point to construct the right BRST operator is the pure spinor condition for λ and $\bar{\lambda}$. Originally, the pure spinor condition was written in term of gamma matrices [26],

$$(\lambda\gamma\lambda)^m = \left(\bar{\lambda}\gamma\bar{\lambda}\right)^m = 0\,, \tag{3.28}$$

which in turns implies

$$\lambda^{\alpha}\lambda^{\beta} = \frac{1}{16\cdot 5!}\gamma^{\alpha\beta}_{mnopq}\left(\lambda\gamma^{mnopq}\lambda\right)\,, \qquad \bar{\lambda}^{\hat{\alpha}}\bar{\lambda}^{\hat{\beta}} = \frac{1}{16\cdot 5!}\gamma^{\hat{\alpha}\hat{\beta}}_{mnopq}\left(\bar{\lambda}\gamma^{mnopq}\bar{\lambda}\right)\,. \tag{3.29}$$

These constraints reduce the elements of λ ($\bar{\lambda}$) from 16 to 11.

The pure spinor constraints in this matrix formulation read

$$\langle\lambda\lambda^*\rangle = 0\,, \qquad \langle\lambda^*\lambda\rangle = 0\,, \tag{3.30}$$

$$\langle\bar{\lambda}\bar{\lambda}^*\rangle = 0\,, \qquad \langle\bar{\lambda}^*\bar{\lambda}\rangle = 0\,. \tag{3.31}$$

One can check that there are actually 5 constraints for $\lambda(\bar{\lambda})$. Therefore,, our ghosts have 11 independent components, as expected. In a similar way to (3.29), (3.30) implies

$${\lambda_a}^{\bar{a}}{\lambda_b}^{\bar{b}} = -\frac{1}{16}\Omega_{ab}\bar{\Omega}^{\bar{a}\bar{b}}\mathrm{tr}\left[\lambda\lambda^*\right] + {\lambda_{(a}}^{\bar{a}}{\lambda_{b)}}^{\bar{b}} + {\lambda_{\langle a}}^{\bar{a}}{\lambda_{b\rangle}}^{\bar{b}}\,, \tag{3.32}$$

and a similar condition for the $\bar{\lambda}$s. Note that

$$\lambda_{(a}{}^{\bar{a}}\lambda_{b)}{}^{\bar{b}} = \lambda_a{}^{(\bar{a}}\lambda_b{}^{\bar{b})} = \lambda_{(a}{}^{(\bar{a}}\lambda_{b)}{}^{\bar{b})}\,, \tag{3.33}$$

and the same is true for $\langle\rangle$.

The ghost Lorentz currents are defined as

$$N_X = \frac{1}{2}\left(\lambda\omega - \omega^*\lambda^*\right)\,, \qquad \bar{N}_X = \frac{1}{2}\left(\bar{\omega}\bar{\lambda} - \bar{\lambda}^*\bar{\omega}^*\right)\,, \tag{3.34}$$

$$N_Y = \frac{1}{2}\left(\omega\lambda - \lambda^*\omega^*\right)\,, \qquad \bar{N}_Y = \frac{1}{2}\left(\bar{\lambda}\bar{\omega} - \bar{\omega}^*\bar{\lambda}^*\right)\,. \tag{3.35}$$

These definitions ensure that the N and $\bar{N}$ terms transform as a gauge term.

Now we can make the identification between the ghost fields in the two descriptions:

$$\omega_\alpha \equiv \omega\,, \qquad \hat{\omega}_{\hat{\alpha}} \equiv \bar{\omega}\,, \qquad \lambda^\alpha \equiv \lambda\,, \qquad \hat{\lambda} \equiv \bar{\lambda}\,, \tag{3.36}$$

$$N^i \equiv \begin{pmatrix} N_X & 0 \\ 0 & N_Y \end{pmatrix} \quad \text{and} \quad \hat{N}^i \equiv \begin{pmatrix} \bar{N}_X & 0 \\ 0 & \bar{N}_Y \end{pmatrix}\,. \tag{3.37}$$

3.3 Symmetries of the AdS Space

The main aim of this article is to write a BRST-invariant superstring action embedded on a $AdS_5 \times S^5$ target space in this formalism of unconstrained matrices. Since such action has to be invariant under the symmetries of a $AdS_5 \times S^5$ space, we first proceed to understand how those symmetries act in this formalism and then we find the structures that are invariant under such symmetries.

3.3.1 Local

A local (gauge) transformation is given by

$$\delta_L Z = ZL + Z \begin{pmatrix} \mathbb{I}\frac{l_X}{4} & 0 \\ 0 & \mathbb{I}\frac{l_Y}{4} \end{pmatrix} , \tag{3.38}$$

where

$$L = \begin{pmatrix} L_X & 0 \\ 0 & L_Y \end{pmatrix} . \tag{3.39}$$

and $\left(L_{X/Y}\right) = L_{X/Y}$. The constraints for the L matrices restrict them to be in $Sp(2) \times Sp(2)$, and the l_X and l_Y are the remaining terms of the stability group.

Thus, a local transformation on the current reads,

$$\delta_L \begin{pmatrix} J_X & K_1 \\ K_3 & J_Y \end{pmatrix} = \begin{pmatrix} [J_X, L_X] + \partial L_X + \mathbb{I}\frac{\partial l_X}{4} & K_1 L_Y - L_X K_1 - K_1 \frac{l_X - l_Y}{4} \\ K_3 L_X - L_Y K_3 + K_3 \frac{l_X - l_Y}{4} & [J_Y, L_Y] + \partial L_Y + \mathbb{I}\frac{\partial l_Y}{4} \end{pmatrix} . \tag{3.40}$$

Using that $L_X K_X = \Omega L_X^T K_X^T \Omega$ we find

$$\delta_L K_X = [K_X, L_X] \, , \qquad \delta_L K_Y = [K_Y, L_Y] \, , \tag{3.41}$$

$$\delta_L A_X = [A_X, L_X] + \partial L_X \, , \qquad \delta_L A_Y = [A_Y, L_Y] + \partial L_Y \, , \tag{3.42}$$

$$\delta_L a_X = \partial l_X \, , \qquad \delta_L a_Y = \partial l_Y \, , \tag{3.43}$$

$$\delta_L F_1 = -L_X F_1 + L_Y F_1 \, , \qquad \delta_L F_3 = -L_Y F_3 - F_3 L_X \, , \tag{3.44}$$

which is expected due to the coset properties.

The first invariant structures that we find are

$$\delta_L \mathrm{tr}\left[K_X \bar{K}_X\right] = \delta_L \mathrm{tr}\left[K_Y \bar{K}_Y\right] = \delta_L \mathrm{tr}\left[K_1 \bar{K}_3\right] = 0 \, . \tag{3.45}$$

The first attempt to construct a Wess-Zumino term will be to use $\left[K_1\bar{K}_1^*\right]$ and $\left[K_3\bar{K}_3^*\right]$. Note that the trace acts on two different spaces. It turns out that those structures are not invariants:

$$\delta_L \ln \mathrm{Tr}\left[K_1\bar{K}_1^*\right] = -\delta_L \ln \mathrm{Tr}\left[K_3\bar{K}_3^*\right] = -2\left(l_X - l_Y\right) . \tag{3.46}$$

To solve this issue we note that $\delta_L E = \left(l_X - l_Y\right) E$. Therefore the right local invariant structures are

$$\delta_L \mathrm{tr}\left[K_1\bar{K}_1^* E^{1/2}\right] = \delta_L \mathrm{tr}\left[K_3\bar{K}_3^* E^{-1/2}\right] = 0\,. \tag{3.47}$$

Since we are equipped with a gauge transformations we can define a covariant derivative,

$$\nabla Z = \partial Z - ZA - Za/4\,, \tag{3.48}$$

where, as expected,

$$A = \begin{pmatrix} A_X & 0 \\ 0 & A_Y \end{pmatrix}, \qquad a = \begin{pmatrix} \mathbb{I}a_X & 0 \\ 0 & \mathbb{I}a_Y \end{pmatrix}, \tag{3.49}$$

and $\left(A_{X/Y}\right) = A_{X/Y}$.

Since $[l, A] = [l, a] = 0$, is straightforward to show

$$\delta_L \nabla Z = \nabla Z \left(L + l/4\right) . \tag{3.50}$$

This is the expected property for the covariant derivative. Finally, just to make everything explicit

$$\nabla Z^{-1} = \partial Z^{-1} + AZ + aZ/4\,, \tag{3.51}$$

$$\nabla E = 0\,. \tag{3.52}$$

The covariant derivative of the global invariant current is

$$\begin{aligned}
\nabla J =& \partial J - \left[J, A + \frac{\mathbb{I}}{4} a\right] \\
=& \begin{pmatrix} \partial J_X & \partial K_1 \\ \partial K_3 & \partial K_Y \end{pmatrix} \\
&+ \begin{pmatrix} [A_X, J_X] & A_X K_1 - K_1 A_Y + \frac{1}{4}(a_X - a_Y) K_1 \\ A_X K_3 - K_3 A_X - \frac{1}{4}(a_X - a_Y) K_3 & [A_Y, J_Y] \end{pmatrix}.
\end{aligned} \tag{3.53}$$

Thus, for the Fs matrices we obtain

$$\nabla F_1 = \partial F_1 + A_X F_1 - F_1 A_Y\,, \tag{3.54}$$

$$\nabla F_3 = \partial F_3 + A_Y F_3 - F_3 A_X\,. \tag{3.55}$$

For the ghosts we require that λ, $\bar{\omega}$ behave as F_1, and $\bar{\lambda}$, ω as F_3. The local invariance of $\mathrm{tr}\left[\omega \bar{\nabla} \lambda\right]$ and $\mathrm{tr}\left[\bar{\omega} \nabla \bar{\lambda}\right]$ requires that

$$\delta_L \lambda = - L_X \lambda + \lambda L_Y\,, \qquad \delta_L \omega = - L_Y \omega + \omega L_X\,, \tag{3.56}$$

$$\delta_L \bar{\lambda} = - L_Y \bar{\lambda} + \bar{\lambda} L_X\,, \qquad \delta_L \bar{\omega} = - L_X \bar{\omega} + \bar{\omega} L_Y\,. \tag{3.57}$$

3.3.2 Global

As stated above, the currents J are invariant under global transformations

$$\delta_G Z = M Z\,, \tag{3.58}$$

where M is any global matrix. The ghosts fields are, by construction, invariant under global transformation, *i.e.*

$$\delta_G(\text{Ghosts}) = 0\,. \tag{3.59}$$

If we compute the global transformation under all the terms constructed above, we find that neither $\left[K_1\bar{K}_1^* E^{1/2}\right]$ nor $\left[K_3\bar{K}_3^* E^{-1/2}\right]$ are invariants for a general M:

$$\delta_G \ln \text{Tr}\left[K_1\bar{K}_1^* E^{1/2}\right] = -\delta_G \ln \text{Tr}\left[K_3\bar{K}_3^* E^{-1/2}\right] = \frac{1}{2}\text{STr}M\,. \tag{3.60}$$

Therefore, we require

$$\text{STr}M = 0\,. \tag{3.61}$$

3.4 BRST transformation and BRST invariant action

In [46] the relation between the GL-formalism with the PSU-formalism constructed in [77] of the Green-Schwarz superstring was established. So far we have established a relation between the elements of the pure spinor string [26] in both the GL-formalism and the PSU-formalism. We have also found all structures invariant under the global and local symmetries of the $AdS_5 \times S^5$ space. In order to construct an action for the pure spinor superstring, we are missing one important ingredient the BRST operator. Below, we will establish the BRST symmetry and then find a BRST invariant action. Before doing so, we will review the BRST symmetry in the PSU-formalism. Then we will construct the BRST symmetry for the GL-formalism and construct the BRST invariant action, using the previous construction as a guide.

3.4.1 PSU-formalism

The BRST transformation for the group element is given by

$$\epsilon\delta_B g = g\epsilon\left(\lambda + \hat{\lambda}\right) . \tag{3.62}$$

When acting on the global invariant current we obtain,

$$\epsilon\delta_B J = \partial\epsilon\left(\lambda + \hat{\lambda}\right) + \left[J, \epsilon\left(\lambda + \hat{\lambda}\right)\right] . \tag{3.63}$$

It is useful to write the transformation for the different $\mathbb{Z}_4$-elements of the current,

$$\epsilon\delta_B J_0 = \left[J_1, \epsilon\hat{\lambda}\right] + \left[J_3, \epsilon\lambda\right] , \tag{3.64}$$

$$\epsilon\delta_B J_1 = \nabla\epsilon\lambda + \left[J_2, \epsilon\hat{\lambda}\right] , \tag{3.65}$$

$$\epsilon\delta_B J_2 = \left[J_1, \epsilon\lambda\right] + \left[J_3, \epsilon\hat{\lambda}\right] , \tag{3.66}$$

$$\epsilon\delta_B J_3 = \nabla\epsilon\hat{\lambda} + \left[J_2, \epsilon\lambda\right] , \tag{3.67}$$

where, as usual, the covariant derivative is defined as $\nabla = \partial + [J_0,]$. The λ and $\hat{\lambda}$ ghosts are invariants under the BRST transformation, but not the ω and $\hat{\omega}$. Thus the BRST transformation for the ghosts is given by,

$$\epsilon\delta_B\omega = -J_3\epsilon, \qquad \epsilon\delta_B\lambda = 0 , \tag{3.68}$$

$$\epsilon\delta_B\hat{\omega} = -\bar{J}_1\epsilon, \qquad \epsilon\delta_B\hat{\lambda} = 0 . \tag{3.69}$$

The ghosts currents were already defined as[2]

$$N = \{\omega, \lambda\} \qquad \text{and} \qquad \hat{N} = \left\{\hat{\omega}, \hat{\lambda}\right\} . \tag{3.70}$$

[2]There is a minus sign of difference between our definition and the definition in [28].

Their BRST transformation are

$$\epsilon\delta_B N = -\left[J_3, \epsilon\lambda\right] \qquad \text{and} \qquad \epsilon\delta_B \hat{N} = -\left[\bar{J}_1, \epsilon\hat{\lambda}\right]. \tag{3.71}$$

In order to prove the BRST invariance of the action we will use the Maurer-Cartan equations. They read

$$\partial\bar{J}_0 - \bar{\partial}J_0 + \left[J_0, \bar{J}_0\right] + \left[J_1, \bar{J}_3\right] + \left[J_2, \bar{J}_2\right] + \left[J_3, \bar{J}_1\right] = 0\,, \tag{3.72a}$$

$$\nabla\bar{J}_1 - \bar{\nabla}J_1 + \left[J_2, \bar{J}_3\right] + \left[J_3, \bar{J}_2\right] = 0\,, \tag{3.72b}$$

$$\nabla\bar{J}_2 - \bar{\nabla}J_2 + \left[J_1, \bar{J}_1\right] + \left[J_3, \bar{J}_3\right] = 0\,, \tag{3.72c}$$

$$\nabla\bar{J}_3 - \bar{\nabla}J_3 + \left[J_1, \bar{J}_2\right] + \left[J_2, \bar{J}_1\right] = 0\,. \tag{3.72d}$$

Now we can show that the action

$$S_{\mathrm{PSU}} = \int d^2z \mathrm{tr}\left[\frac{1}{2}J_2\bar{J}_2 + \frac{1}{4}J_1\bar{J}_3 + \frac{3}{4}\bar{J}_1J_3 + \omega\bar{\nabla}\lambda + \hat{\omega}\nabla\hat{\lambda} - N\hat{N}\right], \tag{3.73}$$

is BRST invariant.

Applying the BRST transformation given by (3.64)-(3.69) to (3.73) we obtain,

$$\begin{aligned}
\epsilon\delta_B S_{\mathrm{PSU}} =& \int d^2z \mathrm{tr}\left\{\frac{1}{2}\left(\left[J_1, \epsilon\lambda\right] + \left[J_3, \epsilon\hat{\lambda}\right]\right)\bar{J}_2 + \frac{1}{2}\left(\left[\bar{J}_1, \epsilon\lambda\right] + \left[\bar{J}_3, \epsilon\hat{\lambda}\right]\right)J_2\right.\\
&+\frac{1}{4}\left(\nabla\epsilon\lambda + \left[J_2, \epsilon\hat{\lambda}\right]\right)\bar{J}_3 + \frac{1}{4}J_1\left(\bar{\nabla}\epsilon\hat{\lambda} + \left[\bar{J}_2, \epsilon\lambda\right]\right) + \frac{3}{4}\left(\bar{\nabla}\epsilon\lambda + \left[\bar{J}_2, \epsilon\hat{\lambda}\right]\right)J_3\\
&+\frac{3}{4}\bar{J}_1\left(\nabla\epsilon\hat{\lambda} + \left[J_2, \epsilon\lambda\right]\right) - J_3\epsilon\bar{\nabla}\lambda - \bar{J}_1\epsilon\nabla\hat{\lambda} + \omega\left[\left[\bar{J}_1, \epsilon\hat{\lambda}\right]\right]\\
&\left.+\left[\left[\bar{J}_3, \epsilon\lambda\right], \lambda\right] + \hat{\omega}\left[\left[J_1, \epsilon\hat{\lambda}\right] + \left[J_3, \epsilon\lambda\right], \hat{\lambda}\right] + \left[J_3, \epsilon\lambda\right]\hat{N} + N\left[\bar{J}_1, \epsilon\hat{\lambda}\right]\right\}\\
=& \int d^2z \mathrm{tr}\left\{\frac{\epsilon\lambda}{4}\left(\bar{\nabla}J_3 - \nabla\bar{J}_3 + \left[\bar{J}_1, J_2\right] - \left[J_1, \bar{J}_2\right]\right)\right.\\
&\left.+\frac{\epsilon\hat{\lambda}}{4}\left(\nabla\bar{J}_1 - \bar{\nabla}J_1 + \left[J_2, \bar{J}_3\right] + \left[J_3, \bar{J}_2\right]\right) - \epsilon\lambda\left[N, \bar{J}_3\right] - \epsilon\hat{\lambda}\left[\hat{N}, J_1\right]\right\}.
\end{aligned} \tag{3.74}$$

Using the pure spinor condition (3.28) and the Maurer-Cartan equations (3.72) in the second equality, we can easily show,

$$\epsilon\delta_B S_{\text{usual}} = 0\,. \tag{3.75}$$

Before ending this section, we note that (3.62) is not actually nilpotent,

$$\epsilon\delta_B\epsilon'\delta_B g = g\epsilon\epsilon'(\lambda\lambda + \bar{\lambda}\bar{\lambda} + \{\lambda, \bar{\lambda}\})\,. \tag{3.76}$$

Using the pure spinor condition (3.28) we can see that $d_B^2 \sim \{\lambda, \bar{\lambda}\}$. Therefore the BRST transformation is nilpotent up to a gauge transformation. The reason for this is that we are ignoring the BRST transformation for the ghosts. It was shown by Chandía in [84] that in a general curved space the pure spinor ghosts acquire a nonvanishing BRST transformation. The case of AdS background was discussed in more detail in [85]. It is straightforward to adapt these results to the present case.

3.4.2 GL-formalism

Now that we are familiar with the original BRST procedure, we can construct the right BRST transformation and the BRST invariant action using a $\frac{GL(4|4)}{(GL(1)\times Sp(2))^2}$ coset. Our ansatz for the BRST transformation of Z is

$$\epsilon\delta_B Z_M{}^A = Z_M{}^B\epsilon\Lambda_B{}^B \qquad \epsilon\delta_B Z_A{}^M = -\epsilon\Lambda_A{}^B Z_B{}^M\,. \tag{3.77}$$

At first one would expect a Λ of the form

$$\Lambda = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix}. \tag{3.78}$$

But a quick computation shows that δ_B^2 is not 0, nor even proportional to a gauge term. A correct form for Λ is

$$\Lambda = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_3 & 0 \end{pmatrix} , \tag{3.79}$$

$$\lambda_1 = \frac{1}{\sqrt{2}} \left(\lambda - \bar{\lambda}^* \right) E^{-1/4} , \tag{3.80}$$

$$\lambda_3 = \frac{1}{\sqrt{2}} \left(\lambda^* + \bar{\lambda} \right) E^{1/4} . \tag{3.81}$$

This is of the right form since we want that $\epsilon\delta_B \ln E = \mathrm{STr}\epsilon\Lambda = 0$, and also that $\delta_B^2 \sim gauge$, as discussed at the end of the previous subsection. Also, the form of λ_1 and λ_3 are such that λ and $\bar{\lambda}$ transform as F_1 and F_3, respectively.

The transformation for the global invariant currents are

$$\epsilon\delta_B K_X = \langle F_1 \epsilon\lambda^* + \epsilon\bar{\lambda}^* F_3 \rangle , \tag{3.82}$$

$$\epsilon\delta_B A_X = \left(F_1 \epsilon\bar{\lambda} - \epsilon\lambda F_3 \right) , \tag{3.83}$$

$$\epsilon\delta_B K_Y = \langle F_1^* \epsilon\lambda + \epsilon\bar{\lambda} F_3^* \rangle , \tag{3.84}$$

$$\epsilon\delta_B A_Y = \left(F_3 \epsilon\lambda - \epsilon\bar{\lambda} F_1 \right) , \tag{3.85}$$

$$\epsilon\delta_B F_1 = \nabla\epsilon\lambda + \epsilon\bar{\lambda}^* K_Y - K_X \epsilon\bar{\lambda}^* , \tag{3.86}$$

$$\epsilon\delta_B F_3 = \nabla\epsilon\bar{\lambda} - \epsilon\lambda^* K_X + K_Y \epsilon\lambda^* , \tag{3.87}$$

and for the ghosts

$$\epsilon\delta_B \omega = -\epsilon F_3 , \qquad \epsilon\delta_B \lambda = 0 , \tag{3.88}$$

$$\epsilon\delta_B \bar{\omega} = \epsilon, \bar{F}_1 , \qquad \epsilon\delta_B \bar{\lambda} = 0 . \tag{3.89}$$

Finally, we are able to show that

$$S_{\mathrm{GL}} = \int d^2 \tilde{\mathrm{Tr}} \left[\frac{1}{2} K_X \bar{K}_X - \frac{1}{2} K_Y \bar{K}_Y + \frac{1}{4} F_1 \bar{F}_3 + \frac{3}{4} \bar{F}_1 F_3 + \omega \bar{\nabla} \lambda + \bar{\omega} \nabla \bar{\lambda} + N_X \bar{N}_X - N_Y \bar{N}_Y \right] \tag{3.90}$$

is BRST invariant. Before we do that, a few comments are in order. $\tilde{\mathrm{Tr}}$ is defined in such a way to avoid confusion on which space the trace acts on. Since tr acts in either a or $\bar{a}$ indices, we cannot write a term like tr $\left(\lambda\omega + \bar{\lambda}\bar{\omega}\right)$. To avoid further confusion, we define an operation $\tilde{\mathrm{Tr}}$ such that $\tilde{\mathrm{Tr}}\left(\lambda\omega + \bar{\lambda}\bar{\omega}\right)$ means $\lambda_a{}^{\bar{a}}\omega_{\bar{a}}{}^a + \bar{\lambda}_{\bar{a}}{}^a\bar{\omega}_a^{\bar{a}}$. Note that the trace of K_Y has a minus sign, that is because $\mathrm{STr}M = M_X - M_Y$. Also, while $\epsilon\delta_B\omega$ has a minus sign, $\epsilon\delta_B\bar{\omega}$ does not. That is because F_3 is related to $-J_3^{\hat{\alpha}}$, and we did that only for aesthetic reasons. Finally, in S_{usual} the ghost current term is tr $-N\hat{N}$, and here is $\tilde{\mathrm{Tr}}N\bar{N}$. The difference in sign is because $\bar{\omega}$ is equivalent to $-\hat{\omega}_{\hat{\alpha}}$. In both actions we want that the kinetic term of the ghost to be positive defined. To obtain that, we need to define $\hat{\omega} = -\hat{\omega}_{\hat{\alpha}}\eta^{\alpha\hat{\alpha}}T_\alpha$, and this in turn implies that tr $-N\hat{N} = N^i\hat{N}^j g_{ij}$ which is equivalent to $\mathrm{STr}N\bar{N}$.

We are going to need the following Maurer-Cartan equation:

$$\bar{\nabla} F_1 - \nabla \bar{F}_1 - \bar{K}_X F_3^* + K_X \bar{F}_3^* - \bar{F}_3^* K_Y + F_3^* \bar{K}_Y = 0, \tag{3.91}$$

$$\bar{\nabla} F_3 - \nabla \bar{F}_3 + \bar{K}_Y F_1^* - K_Y \bar{F}_1^* + \bar{F}_1^* K_X - F_1^* \bar{K}_X = 0. \tag{3.92}$$

We now check that

Applying the BRST transformation to (3.90),

$$\begin{aligned}\epsilon\delta_B S_{\mathrm{RS}} = \int d^2z\tilde{\mathrm{Tr}}\Big\{&\frac{1}{2}\left(F_1\epsilon\lambda^* + \epsilon\bar{\lambda}^* F_3\right)\bar{K}_X + \frac{1}{2}K_X\left(\bar{F}_1\epsilon\lambda^* + \epsilon\bar{\lambda}^*\bar{F}_3\right)\\ &-\frac{1}{2}\left(F_1^*\epsilon\lambda + \epsilon\bar{\lambda}F_3^*\right)\bar{K}_Y - \frac{1}{2}K_Y\left(\bar{F}_1^*\epsilon\lambda + \epsilon\bar{\lambda}\bar{F}_3^*\right) + \frac{1}{4}\left(\nabla\epsilon\lambda + \epsilon\bar{\lambda}^* K_Y - K_X\epsilon\bar{\lambda}^*\right)\bar{F}_3\\ &+\frac{1}{4}F_1\left(\bar{\partial}\epsilon\bar{\lambda} - \epsilon\lambda^*\bar{K}_X + \bar{K}_Y\epsilon\lambda^*\right) + \frac{3}{4}\left(\bar{\nabla}\epsilon\lambda + \epsilon\bar{\lambda}^*\bar{K}_Y - \bar{K}_X\epsilon\bar{\lambda}^*\right)F_3\\ &+\frac{1}{4}\bar{F}_1\left(\partial\epsilon\bar{\lambda} - \epsilon\lambda^* K_X + K_Y\epsilon\lambda^*\right) - \epsilon F_3\bar{\nabla}\lambda + \epsilon\bar{F}_1\nabla\bar{\lambda} + \left(\bar{F}_1\epsilon\bar{\lambda} - \epsilon\lambda\bar{F}_3\right)N_X\\ &-\left(F_1\epsilon\bar{\lambda} - \epsilon\lambda F_3\right)\bar{N}_X - \left(\bar{F}_3\epsilon\lambda - \epsilon\bar{\lambda}\bar{F}_1\right)N_Y + \left(F_3\epsilon\lambda - \epsilon\bar{\lambda}F_1\right)\bar{N}_Y\\ &-\epsilon\lambda F_3\bar{N}_X + N_X\epsilon\bar{F}_1\bar{\lambda} + \epsilon F_3\lambda\bar{N}_Y - N_Y\bar{\lambda}\epsilon\bar{F}_1\Big\}\,.\end{aligned} \tag{3.93}$$

The pure spinor condition ensures that $N_X\lambda - \lambda N_Y = 0$ and $\bar{N}_Y\bar{\lambda} - \bar{\lambda}\bar{N}_X = 0$. The only terms that survive are

$$\epsilon\delta_B S_{\mathrm{RS}} = \int d^2z\frac{1}{4}\tilde{\mathrm{Tr}}\left[\epsilon\bar{\lambda}\left(\bar{\nabla}F_1 - \nabla\bar{F}_1 - \bar{K}_X F_3^* + K_X\bar{F}_3^* - \bar{F}_3^* K_Y + F_3^*\bar{K}_Y\right)\right. \tag{3.94}$$

$$\left.+\epsilon\lambda\left(\bar{\nabla}F_3 - \nabla\bar{F}_3 + \bar{K}_Y F_1^* - K_Y\bar{F}_1^* + \bar{F}_1^* K_X - F_1^*\bar{K}_X\right)\right]\,, \tag{3.95}$$

which are identically 0 because of the Maurer-Cartan equation,

$$S_{WZ} = -\frac{1}{4}\int d^2z\mathrm{tr}\left[K_3^*\bar{K}_3 E^{-1/2} - K_1\bar{K}_1^* E^{1/2}\right] = -\frac{1}{4}\int d^2z\mathrm{tr}\left[F_1\bar{F}_3 - \bar{F}_1 F_3\right]\,. \tag{3.96}$$

As we saw in the previous section, (3.90) is both local and global invariant if and only if the global transformation is generated by a supertraceless matrix.

3.4.3 Vectors

In [75, 74] a systematic construction of vertex operators for a supersphere sigma model was developed. An important ingredient for such construction was vectors describing the target spaced. The existence of such vectors describing the bosonic coordinates of the $Ads_5 \times S^5$ superspace was discussed in [46]. It is an interesting question whether we can construct all the matter part of (3.90) with such vectors. We will now discuss how to obtain this. The first set of vectors we can construct are

$$W^{MN} = Z_a{}^M \Omega^{ab} Z_b{}^N E^{1/4}\,, \qquad W_{MN} = Z_M{}^a \Omega_{ab} Z_N{}^b E^{-1/4}\,, \tag{3.97}$$

$$W'^{MN} = Z_{\bar{a}}{}^M \Omega^{\bar{a}\bar{b}} Z_{\bar{b}}{}^N E^{-1/4}\,, \qquad W'_{MN} = Z_M{}^{\bar{a}} \Omega_{\bar{a}\bar{b}} Z_N{}^{\bar{b}} E^{1/4}\,. \tag{3.98}$$

Being careful with the indices and product of fermionic matrices, the only terms that we can construct are,

$$\nabla W^{MN} \bar{\nabla} W_{NM} = \mathrm{tr}\left[4K_X \bar{K}_X + 2K_1 \bar{K}_3\right]\,, \tag{3.99}$$

$$\nabla W'^{MN} \bar{\nabla} W'_{NM} = \mathrm{tr}\left[4K_Y \bar{K}_Y + 2K_3 \bar{K}_1\right]\,. \tag{3.100}$$

Now we are able to construct part of the matter part of (3.90),

$$\frac{1}{8}\left[\nabla W^{MN} \bar{\nabla} W_{NM} - \nabla W'^{MN} \bar{\nabla} W'_{NM}\right] = \tilde{\mathrm{T}}\mathrm{r}\left[\frac{1}{2}K_X \bar{K}_X - \frac{1}{2}K_Y \bar{K}_Y + \frac{1}{4}K_1 \bar{K}_3 + \frac{1}{4}\bar{K}_1 K_3\right]\,. \tag{3.101}$$

In order to obtain the right factor for the K-terms, we need to introduce another group of vectors,

$$U^{MN} = Z_a{}^M \Omega^{ab} \overleftrightarrow{\nabla} Z_b{}^N E^{1/4}\,, \qquad U_{MN} = Z_M{}^a \Omega_{ab} \overleftrightarrow{\nabla} Z_N{}^b E^{-1/4}\,, \tag{3.102}$$

$$U'^{MN} = Z_{\bar{a}}{}^M \Omega^{\bar{a}\bar{b}} \overleftrightarrow{\nabla} Z_{\bar{b}}{}^N E^{-1/4}\,, \qquad U'_{MN} = Z_M{}^{\bar{a}} \Omega_{\bar{a}\bar{b}} \overleftrightarrow{\nabla} Z_N{}^{\bar{b}} E^{1/4}\,. \tag{3.103}$$

We define $A \overleftrightarrow{\nabla} B = A\nabla B - \nabla A B$, and ∇ is the covariant derivative defined in (3.48). A direct computation shows that the product (in this case, the STr) of any two different vectors is always 0.

Using (3.99) and (3.100) we can construct,

$$U^{MN}\bar{U}_{NM} = \mathrm{tr}\left[-2K_1\bar{K}_3\right] , \tag{3.104}$$

$$\bar{U}'^{MN}U'_{MN} = \mathrm{tr}\left[-2K_3\bar{K}_1\right] . \tag{3.105}$$

With all those ingredients, we can construct the matter part of (3.90) without the Wess-Zumino term:

$$\frac{1}{8}\left[\nabla W^{MN}\bar{\nabla}W_{NM} - U^{MN}\bar{U}_{NM} - \nabla W'^{MN}\bar{\nabla}W'_{NM} + U'^{MN}\bar{U}'_{NM}\right] \tag{3.106}$$

$$= \frac{1}{2}\tilde{\mathrm{Tr}}\left[K_X\bar{K}_X - K_Y\bar{K}_Y + K_1\bar{K}_3 + \bar{K}_1K_3\right] . \tag{3.107}$$

The question now is how can we write the Wess-Zumino term of the action. First we remember that the Wess-Zumino term is

$$\mathcal{L}_{\mathrm{WZ}} = -\frac{\kappa}{2}\mathrm{tr}\left[F_1\bar{F}_3 - \bar{F}_1F_3\right] = \frac{\kappa}{2}\mathrm{tr}\left[K_1\bar{K}_1^*E^{1/2} - K_3^*\bar{K}_3E^{-1/2}\right] . \tag{3.108}$$

A quick glance to list of vectors shows that the only possible way is a product between Ws and Us. Indeed

$$(-)^M\nabla W^{MN}\bar{U}'_{NM} = \mathrm{tr}\left[-2K_1\bar{K}_1^*E^{1/2}\right] , \tag{3.109}$$

$$(-)^M\nabla W'^{MN}\bar{U}_{NM} = \mathrm{tr}\left[-2K_3\bar{K}_3^*E^{-1/2}\right] . \tag{3.110}$$

Before we continue, a comment should be made: The product between vector is a STr between supermatrices,

$$W^{MN}W_{NM} = \mathrm{STr}WW. \tag{3.111}$$

The product between W and U' should also be a STr. The product $\nabla W^{MN}\bar{U}'_{NM}$ is not, since

$$\nabla W^{MN}\bar{U}'_{NM} \neq \bar{U}'_{NM}\nabla W^{MN} \,. \tag{3.112}$$

The solution to this problem is the addition of the $(-)^{M}$ term. Now

$$(-)^{M}\nabla W^{MN}\bar{U}'_{NM} = \mathrm{STr}\nabla W\bar{U}' \,. \tag{3.113}$$

We finally have all the ingredients to construct the matter part of S_{GL} and choosing $\kappa = \frac{1}{2}$, we get

$$\begin{aligned} \mathcal{L}_{\mathrm{RS}} =& \frac{1}{8}\left[\nabla W^{MN}\bar{\nabla}W_{NM} - U^{MN}\bar{U}_{NM} - (-)^{M}\nabla W^{MN}\bar{U}'_{NM} - \nabla W'^{MN}\bar{\nabla}W'_{NM}\right. \\ & \left. + U'^{MN}\bar{U}'_{NM} + (-)^{M}\nabla W'^{MN}\bar{U}_{NM}\right] \qquad (3.114) \\ =& \frac{1}{2}\tilde{\mathrm{T}}\mathrm{r}\left[K_X\bar{K}_X - K_Y\bar{K}_Y + \frac{1}{2}K_1\bar{K}_3 + \frac{3}{2}\bar{K}_1K_3\right] \,. \qquad (3.115) \end{aligned}$$

3.5 An application: vertex operator construction

Following [69, 82] we will construct an operator V such that $\epsilon\delta_B V = 0$. To achieve this, we will construct the conserved current j related to global symmetries of the action (3.90). Then we will construct V by applied the BRST transformation to j, $\delta_B j = \partial V$. This will be our first vertex operator in this formalism. In future works we will try to apply the procedure explained in sections 4.3 to the construction of vertex operators, as in [75].

3.5.1 Equation of Motions

As usual, in order to construct a conserved current, we need the equation of motions (EOM). To obtains such equations we will vary Z around a background field,

$$Z = Z_0 e^X \,, \tag{3.116}$$

where the components of X have been defined in (3.4). This leads to

$$\delta J = \partial X + [J, X] \,. \tag{3.117}$$

Writing this in components

$$\delta J_X = \partial X + [J_X, X] + K_1 \Theta_3 - \Theta_1 K_3 \,, \tag{3.118a}$$

$$\delta J_Y = \partial Y + [J_Y, Y] + K_3 \Theta_1 - \Theta_3 K_1 \,, \tag{3.118b}$$

$$\delta K_1 = \nabla \Theta_1 + K_X \Theta_1 - \Theta_1 K_Y + K_1 Y - X K_1 \,, \tag{3.118c}$$

$$\delta K_3 = \nabla \Theta_3 + K_Y \Theta_3 - \Theta_3 K_X + K_3 X - Y K_3 \,. \tag{3.118d}$$

Since we have written (3.90) in terms of F_1 and F_3, we write the variation of those, using the above equations:

$$\delta F_1 = \nabla \theta_1 - K_X \theta_3^* + \theta_3^* K_Y - F_3^* Y + X F_3^* \,, \tag{3.119}$$

$$\delta F_3 = \nabla \theta_3 + K_Y \theta_1^* - \theta_1^* K_X + F_1^* X - Y F_1^* \,, \tag{3.120}$$

and the same for the Ks and As:

$$\delta K_X = \nabla X + \frac{1}{2} \left(F_1 \theta_1^* - \theta_1 F_1^* + \theta_3^* F_3 - F_3^* \theta_3 \right) - \frac{\mathbb{I}}{4} \mathrm{tr} \left(F_1 \theta_1^* + \theta_3 * F_3 \right) \,, \tag{3.121}$$

$$\delta K_Y = \nabla Y + \frac{1}{2} \left(F_1^* \theta_1 - \theta_1^* F_1 + \theta_3 F_3^* - F_3 \theta_3^* \right) - \frac{\mathbb{I}}{4} \mathrm{tr} \left(F_1 \theta_1^* + \theta_3 * F_3 \right) \,, \tag{3.122}$$

$$\delta A_X = [K_X, X] + \frac{1}{2}\left(F_1\theta_3 + \theta_3^* F_1^* - \theta_1 F_3 - F_3^*\theta_1^*\right)\,, \tag{3.123}$$
$$\delta A_Y = [K_Y, Y] + \frac{1}{2}\left(F_3\theta_1 + \theta_1^* F_3^* - \theta_3 F_1 - F_1^*\theta_3^*\right)\,. \tag{3.124}$$

Using the variation of the action and the Maurer-Cartan equations we obtain,

$$\nabla\bar{K}_X + \frac{1}{2}\left(F_1\bar{F}_1^* - \bar{F}_1F_1^*\right) - \frac{\mathbb{I}}{4}\mathrm{tr}\,F_1\bar{F}_1^* + \left[\bar{N}_X, K_X\right] - \left[N_X, \bar{K}_X\right] = 0\,, \tag{3.125a}$$
$$\bar{\nabla}K_X + \frac{1}{2}\left(F_3^*\bar{F}_3 - \bar{F}_3^*F_3\right) - \frac{\mathbb{I}}{4}\mathrm{tr}\,F_3\bar{F}_3^* + \left[\bar{N}_X, K_X\right] - \left[N_X, \bar{K}_X\right] = 0\,, \tag{3.125b}$$
$$\nabla\bar{K}_Y + \frac{1}{2}\left(F_1^*\bar{F}_1 - \bar{F}_1^*F_1\right) - \frac{\mathbb{I}}{4}\mathrm{tr}\,F_1\bar{F}_1^* + \left[\bar{N}_Y, K_Y\right] - \left[N_Y, \bar{K}_Y\right] = 0\,, \tag{3.125c}$$
$$\bar{\nabla}K_Y + \frac{1}{2}\left(F_3\bar{F}_3^* - \bar{F}_3F_3^*\right) - \frac{\mathbb{I}}{4}\mathrm{tr}\,F_3\bar{F}_3^* + \left[\bar{N}_Y, K_Y\right] - \left[N_Y, \bar{K}_Y\right] = 0\,, \tag{3.125d}$$
$$\bar{\nabla}\bar{F}_1 - \bar{K}_XF_3^* + K_X\bar{F}_3^* + F_3^*\bar{K}_Y - \bar{F}_3^*K_Y - N_X\bar{F}_1 + \bar{N}_XF_1 + \bar{F}_1N_Y - F_1\bar{N}_Y = 0\,, \tag{3.125e}$$
$$\nabla F_1 - N_X\bar{F}_1 + \bar{N}_XF_1 + \bar{F}_1N_Y - F_1\bar{N}_Y = 0\,, \tag{3.125f}$$
$$\nabla\bar{F}_3 - \bar{K}_Y^*F_1 + K_Y^*\bar{F}_1 + F_1^*\bar{K}_X - \bar{F}_1^*K_X - N_Y\bar{F}_3 + \bar{N}_YF_3 + \bar{N}_YF_3 - F_3\bar{N}_X = 0\,, \tag{3.125g}$$
$$\bar{\nabla}F_3 - N_Y\bar{F}_3 + \bar{N}_YF_3 + \bar{F}_3N_X - F_3\bar{N}_X = 0\,, \tag{3.125h}$$
$$\bar{\nabla}\omega + \omega\bar{N}_X - \bar{N}_Y\omega = 0\,, \tag{3.125i}$$
$$\bar{\nabla}\lambda + \bar{N}_X\lambda - \lambda\bar{N}_Y = 0\,, \tag{3.125j}$$
$$\nabla\bar{\omega} - N_X\bar{\omega} + \bar{\omega}N_Y = 0\,, \tag{3.125k}$$
$$\nabla\bar{\lambda} + \bar{\lambda}N_X - N_Y\bar{\lambda} = 0\,. \tag{3.125l}$$

To obtain these equations we used the fact $\mathrm{tr}\,[XH] = \mathrm{tr}\,[\langle X\rangle H] = \mathrm{tr}\,[X\langle H\rangle]$. Thus, the right EOM for X is given by $\langle H\rangle = 0$.

3.5.2 Construction of V

In order to compute the Noether current we first make a few observations. The first of them is noting that $\mathrm{tr}K_X\bar{K}_X = \mathrm{tr}K_X\bar{J}_X$, thus, instead of taking $\mathrm{tr}\bar{K}_X\langle Z_a{}^M\partial M_M{}^N Z_N{}^b\rangle$, we just take $\mathrm{tr}\bar{K}_X Z_a{}^M\partial M_M{}^N Z_N{}^b$. The same can be done for the ghost current, since $N_X = (\lambda\omega)$.

Using the EOM (3.125) and the global transformation studied in section 3.2, we find the left and right conserved currents,

$$j = \begin{pmatrix} K_X + 2N_X & \frac{1}{2\sqrt{2}}\left(F_1 - 3F_3^*\right)E^{-1/4} \\ \frac{1}{2\sqrt{2}}\left(F_1^* + 3F_3^*\right)E^{1/4} & K_Y + 2N_Y \end{pmatrix}, \tag{3.126}$$

$$\bar{j} = \begin{pmatrix} \bar{K}_X - 2\bar{N}_X & \frac{1}{2\sqrt{2}}\left(3\bar{F}_1 - \bar{F}_3^*\right)E^{-1/4} \\ \frac{1}{2\sqrt{2}}\left(3\bar{F}_1^* + \bar{F}_3^*\right)E^{1/4} & \bar{K}_Y - 2\bar{N}_Y \end{pmatrix}. \tag{3.127}$$

Since $\epsilon\delta_B\delta_G S_{\rm RS} = 0$ one would expect $\epsilon\delta_B j = \partial V$ and $\epsilon\delta_B \bar{j} = -\bar{\partial} V$ as in the usual description. But here $\mathrm{STr} M = 0$, thus, $\epsilon\delta_B j = \partial V + \mathbb{I}A$ and $\epsilon\delta_B\bar{j} = -\bar{\partial}V + \mathbb{I}B$ is the most general form, for any A and B. For the same reason, one would expect that $\epsilon\delta_B\epsilon'\delta_B\delta_G S = 0$ yields $\epsilon\delta_B V = 0$, but the most general possibility is $\epsilon\delta_B V = \mathbb{I}C$, for any C. Now, this $\mathbb{I}C$ term should be expected from the gauge group $(GL(1))^2$, since a the condition $A = \Omega A^T\Omega$, imposed to gauge terms, does not apply to the term proportional to the trace[3] thus, it seems that we have eliminated those term. But this is not true, we did eliminated the a_X, a_Y gauge terms: we did it when writing the action proportional to the tr. Therefore, the correct BRST invariant vector is $\mathrm{STr}V$.

After a long calculation, for BRST transformation of the left current we find that

$$\epsilon\delta_B j = \frac{1}{2\sqrt{2}}\partial\epsilon V - \frac{\mathbb{I}}{4}\mathrm{tr}\left(F_1\epsilon\lambda^* + \epsilon\bar{\lambda}^* F_3\right), \tag{3.128}$$

$$\epsilon V = Z\begin{pmatrix} 0 & \epsilon\left(\lambda + \bar{\lambda}^*\right)E^{-1/4} \\ \epsilon\left(\lambda^* - \bar{\lambda}\right)E^{1/4} & 0 \end{pmatrix} Z^{-1} = Z\epsilon\Lambda' Z^{-1}. \tag{3.129}$$

For the right current we find, as expected,

$$\epsilon\delta_B\bar{j} = -\frac{1}{2\sqrt{2}}\bar{\partial}\epsilon V - \frac{\mathbb{I}}{4}\mathrm{tr}\left(\bar{F}_1\epsilon\lambda^* + \epsilon\bar{\lambda}^*\bar{F}_3\right). \tag{3.130}$$

[3] Note that $\mathbb{I} = -\Omega\mathbb{I}^T\Omega$.

Finally, we check that $\epsilon\delta_B \mathrm{STr} V = 0$:

$$\epsilon'\delta_B\epsilon V = Z\left[\epsilon'\Lambda, \epsilon\Lambda'\right] Z^{-1} = Z\epsilon'\epsilon\left\{\Lambda, \Lambda'\right\} Z^{-1} \tag{3.131}$$

$$= 2\epsilon'\epsilon Z \begin{pmatrix} \lambda\lambda^* + \bar{\lambda}^*\bar{\lambda} & 0 \\ 0 & \lambda^*\lambda + \bar{\lambda}\bar{\lambda}^* \end{pmatrix} Z^{-1} \tag{3.132}$$

$$= \frac{1}{2}\epsilon'\epsilon\mathbb{I}\mathrm{tr}\left(\lambda\lambda^* + \bar{\lambda}^*\bar{\lambda}\right), \tag{3.133}$$

therefore $\delta_B \mathrm{STr} V = 0$. The vertex operator corresponding to the β-deformation discussed in [69, 82] can now be described as the tensor product of two V.

3.6 Conclusion and further directions

We have described the pure spinor superstring in $AdS_5 \times S^5$ using the $GL(4|4)/(Sp(2) \times GL(1))^2$ coset first used by Roiban and Siegel for the Green-Schwarz superstring in [46]. This formulation provides additional choices for the parametrization of the AdS coordinates. This additional choices have been shown to be useful in formulations different superspaces relevant to the AdS/CFT conjecture [83]. Recently, Schwarz described another parametrization for the GS string in $AdS_5 \times S^5$ [86]. As was shown by Siegel [87], this new formulation can also be used in the present case.

Furthermore, the complete superspace propagator for the entire tower of Kaluza-Klein modes was calculated in [30] using this new coset. This propagator was shown to be invariant under κ-symmetry. Since there is a close relation between κ-symmetry and the BRST transformations of the pure spinor formalism [4] it is likely that this propagator can be used to construct a BRST invariant ghost number two superspace function. Such function would be related to the unintegrated vertex operators of the supergravity modes in the pure spinor formulation. We are presently working in this direction. The ultimate goal is to have a

[4] For example, demanding invariance under κ-symmetry of the GS action in a general curved supergravity background puts the background on-shell. The same is achieved in pure spinor formalism demanding BRST invariance [88].

systematic way to construct vertex operators at any mass level using the world sheet dilatation operator [43] to derive physical state conditions. Although BRST invariance should also be imposed, vanishing world sheet anomalous dimension may be enough to calculate the spacetime energy of the string states.

Chapter 4

Stress-tensor OPE in $\mathcal{N} = 2$ superconformal theories

4.1 Introduction

Four-dimensional superconformal field theories (SCFTs) with $\mathcal{N} = 2$ supersymmetry play a prominent role in theoretical physics. Originally studied using standard field theoretic tools, by building Lagrangians out of fundamental fields with appropriately chosen matter content, the list of theories has grown considerable in recent years [39, 40], and now there seems to be an extensive library of $\mathcal{N} = 2$ systems, related by an intricate web of dualities. Having found such an ample catalog, there has been a shift in perspective, instead of analyzing specific models one by one, it seems more natural to ask whether a classification program is possible. Efforts in this direction include a classification of Lagrangian models [41], a procedure for classifying class $\mathcal{S}$ theories [89], and a systematic analysis of Coulomb branch geometries [90, 91].

Among the most important tools for constraining the space of CFTs is the conformal bootstrap approach [5, 4, 6]. Originally very successful in two dimensions, where the conformal algebra is enhanced to the infinite dimensional Virasoro algebra, it has seen renewed interest in light of the work of [9] where, starting from basic principles like crossing symmetry and unitarity, numerical techniques were developed that allow to obtain rigorous bounds on several CFT quantities. Influenced by this revival of the bootstrap philosophy, the $\mathcal{N} = 2$ superconformal bootstrap program was initiated in [92, 42], with the goal of serving as an organizing principle, relying only on the operator algebra of a theory, as defined by the OPE.

The $\mathcal{N} = 2$ superconformal bootstrap program can be thought of as a two-step process. First, it was observed in [92] that any $\mathcal{N} = 2$ SCFT contains a protected subsector of observables described by a two-dimensional chiral algebra. In order to bootstrap a full-fledged SCFT, one must first have an understanding of the operators described by the chiral algebra. Once this is achieved, the second step entails tackling the harder task of bootstrapping the full theory, in particular, unprotected operators with unconstrained conformal dimensions. This second step was explored in [42] using the numerical techniques of [9], and bounds were obtained by looking at four-point correlators of several superconformal multiplets. Though a comprehensive effort toward bootstrapping the landscape of $\mathcal{N} = 2$ theories, there was an important omission, the multiplet in which the stress-tensor sits was absent from the analysis. The universal nature of the stress-tensor makes it a natural target for bootstrap studies, and the reason it was not included in [42] was technical: the requisite crossing symmetry equation is not known.

Let us be a bit more specific. The conserved stress-tensor of an $\mathcal{N} = 2$ theory sits in a multiplet that can be represented by a superfield $\mathcal{J}$ with a schematic θ-expansion,

$$\mathcal{J}(x,\theta,\bar\theta)| = J(x)\,, \qquad \mathcal{J}(x,\theta,\bar\theta)|_{\theta\bar\theta} = J^{ij}_{\mu}(x)\,, \qquad \mathcal{J}(x,\theta,\bar\theta)|_{\theta^2\bar\theta^2} = T_{\mu\nu}(x)\,. \tag{4.1}$$

J is a scalar superconformal primary of dimension two, J^{ij}_{μ} is the conserved $SU(2)_R \times U(1)_r$ R-symmetry current, and $T_{\mu\nu}$ is the stress-tensor. Correlators of this multiplet also contain information about two fundamental quantities present in any four-dimensional CFT, the a and c anomaly coefficients. These can be defined as the anomalous trace of the stress-tensor when the theory is considered in a curved background.

Our first goal will be to obtain the supersymmetric selection rules

$$\mathcal{J} \times \mathcal{J} \sim \ldots\,, \tag{4.2}$$

namely, the $\mathcal{N}=2$ multiplets that are allowed to appear in the super OPE of two stress-tensor multiplets. This result will be relevant for both the two-dimensional chiral algebra description and the numerical bounds program.

On the chiral algebra side, as observed in [92], the two-dimensional stress-tensor can be associated to the four-dimensional $SU(2)_R$ current. In particular, correlators of the four-dimensional current have a solvable truncation described by correlators of the two-dimensional holomorphic stress-tensor. The holomorphic correlator only depends on the central charge c and, as we will see in this work, unitarity of the four-dimensional theory implies an analytic bound on c. The a anomaly coefficient plays no role in the chiral algebra construction.

On the numerical side, the super OPE selection rules are the first step toward writing the crossing symmetry equation for the stress-tensor multiplet. To have a better understanding of how this can be accomplished, let us recall how the numerical bootstrap is implemented. The starting point is the four-point function of a real scalar operator ϕ. This correlator can be expanded using a *conformal block* expansion, where each conformal block captures the contribution of a specific conformal family appearing in the $\phi\times\phi$ OPE. Explicit expressions for scalar conformal blocks were obtained in [93, 94]. Having obtained such an expansion, using the restrictions imposed by crossing symmetry and unitarity, it is possible to obtain numerical bounds on scaling dimensions and three-point couplings. In $\mathcal{N}=2$ theories, the highest weight of the stress-tensor multiplet is a scalar of dimension $\Delta_J = 2$, and is therefore well suited for the numerical bootstrap program. Because of supersymmetry, several conformal families are related by the action of supercharges, and this implies that a finite number of conformal blocks appearing in a correlator can be grouped together in a *superconformal block*, which encodes the contribution of the corresponding superconformal family. We can now state more precisely why the stress-tensor correlator was not included in [42]: the superconformal block expansion of the $\langle JJJJ \rangle$ correlator has not been worked out. To fill this gap in the $\mathcal{N} = 2$ literature was one of the motivations for this work.

Conformal and superconformal block expansions are a common obstacle in any attempt to write bootstrap equations. In the bosonic case, things get very complicated when one considers operators in non-trivial Lorentz representations. With supersymmetry, many complications arise for correlators of generic multiplets. There is no unifying framework and a wide variety of approaches have been tried with varying degrees of success [95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108]. The full superconformal block expression for the J correlator is still elusive, and it is not clear which of all the methods available in the literature is the most efficient. Nevertheless, our calculation encodes the allowed $\mathcal{N} = 2$ multiplets that contribute to the expansion, which is the first step toward writing the crossing symmetry equation.

The outline of the paper is as follows. In section 2 we review the conformal algebra and its shortening conditions. Section 3 presents a detailed superspace analysis that allows us to write the super OPE selection rules for two stress-tensor multiplets. In section 4 we use our selection rules together with the two-dimensional chiral algebra construction to obtain an analytic bound on c. This bound is valid for any $\mathcal{N} = 2$ superconformal theory regardless of its matter content and flavor symmetries. In section 5 we present a partial analysis of the superconformal block expansion of the J correlator.

4.2 Preliminaries

The $\mathcal{N} = 2$ superconformal algebra is the algebra of the supergroup $SU(2, 2|2)$. It contains the conformal algebra $SU(2, 2) \sim SO(4, 2)$ with generators $\{\mathcal{P}_{\alpha\dot{\alpha}}, \mathcal{K}^{\dot{\alpha}\alpha}, \mathcal{M}_{\alpha}{}^{\beta}, \bar{\mathcal{M}}^{\dot{\alpha}}{}_{\dot{\beta}}, D\}$, where $\alpha = \pm$ and $\dot{\alpha} = \pm$ are Lorentz indices, and an $SU(2)_R \times U(1)_r$ R-symmetry algebra with generators $\{\mathcal{R}^i{}_j, r\}$, where $i = 1, 2$ are $SU(2)_R$ indices. In addition to the bosonic generators there are fermionic supercharges, the Poincaré and conformal supercharges, $\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}_{i\,\dot\alpha}, \mathcal{S}^\alpha_i, \bar{\mathcal{S}}^{i\,\dot\alpha}\}$.

A general supermultiplet of $SU(2,2|2)$ contains a highest weight or superconformal primary with quantum number $(\Delta, j, \bar\jmath, R, r)$, where $(\Delta, j, \bar\jmath)$ are the Dynkin labels of the conformal group and (R, r) the Dynkin labels of the R-symmetry. The highest weight is, by definition, annihilated by the supercharges $\mathcal{S}$ and $\bar{\mathcal{S}}$ and the multiplet is then constructed by successive action of the Poincaré supercharges. Generic supermultiplets are called long multiplets and we will denote them by $\mathcal{A}^\Delta_{R,r(j,\bar\jmath)}$ following the conventions of [109]. Unitarity imposes restrictions on the conformal dimension of $\mathcal{A}$ known as unitarity bounds. For generic long multiplets the bounds read,

$$\Delta \geq 2 + 2j + 2R + r\,,\; 2 + 2\bar\jmath + 2R - r\,. \tag{4.3}$$

If the highest weight is annihilated by some combination of the supercharges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ the multiplet shortens. There are several types of shortening conditions depending on the Lorentz and $SU(2)_R$ quantum numbers of the charges that kill the highest weight, we denote them $\mathcal{B}$-type and $\mathcal{C}$-type shortening conditions.

$$\mathcal{B}^i \quad : \quad \mathcal{Q}^i_\alpha \Psi = 0, \tag{4.4}$$

$$\overline{\mathcal{B}}_i \quad : \quad \bar{\mathcal{Q}}_{i\dot\alpha} \Psi = 0, \tag{4.5}$$

$$\mathcal{C}^i \quad : \quad \begin{cases} \varepsilon^{\alpha\beta} \mathcal{Q}^i_\alpha \Psi_\beta = 0, & j \neq 0 \\ \varepsilon^{\alpha\beta} \mathcal{Q}^i_\alpha \mathcal{Q}^i_\beta \Psi = 0, & j = 0 \end{cases} \tag{4.6}$$

$$\overline{\mathcal{C}}_i \quad : \quad \begin{cases} \varepsilon^{\dot\alpha\dot\beta} \bar{\mathcal{Q}}_{i\dot\alpha} \Psi_{\dot\beta} = 0, & \bar\jmath \neq 0 \\ \varepsilon^{\dot\alpha\dot\beta} \bar{\mathcal{Q}}_{i\dot\alpha} \bar{\mathcal{Q}}_{i\dot\beta} \Psi = 0, & \bar\jmath = 0 \end{cases} \tag{4.7}$$

$\mathcal{B}$-type conditions are sometimes called short while $\mathcal{C}$-type are sometimes called semi-short. In table 4.1 we present all possible shortening conditions for the $\mathcal{N} = 2$ superconformal algebra following the notation of [109]. Among the most important short multiplets are the

Shortening	Quantum Number Relations		Multiplet
$\mathcal{B}^1$	$\Delta = 2R + r$	$j = 0$	$\mathcal{B}_{R,r(0,\bar{\jmath})}$
$\bar{\mathcal{B}}_2$	$\Delta = 2R - r$	$\bar{\jmath} = 0$	$\bar{\mathcal{B}}_{R,r(j,0)}$
$\mathcal{B}^1 \cap \mathcal{B}^2$	$\Delta = r$	$R = 0$	$\mathcal{E}_{r(0,\bar{\jmath})}$
$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$\Delta = -r$	$R = 0$	$\bar{\mathcal{E}}_{r(j,0)}$
$\mathcal{B}^1 \cap \bar{\mathcal{B}}_2$	$\Delta = 2R$	$j = \bar{\jmath} = r = 0$	$\hat{\mathcal{B}}_R$
$\mathcal{C}^1$	$\Delta = 2 + 2j + 2R + r$		$\mathcal{C}_{R,r(j,\bar{\jmath})}$
$\bar{\mathcal{C}}_2$	$\Delta = 2 + 2\bar{\jmath} + 2R - r$		$\bar{\mathcal{C}}_{R,r(j,\bar{\jmath})}$
$\mathcal{C}^1 \cap \mathcal{C}^2$	$\Delta = 2 + 2j + r$	$R = 0$	$\mathcal{C}_{0,r(j,\bar{\jmath})}$
$\bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$\Delta = 2 + 2\bar{\jmath} - r$	$R = 0$	$\bar{\mathcal{C}}_{0,r(j,\bar{\jmath})}$
$\mathcal{C}^1 \cap \bar{\mathcal{C}}_2$	$\Delta = 2 + 2R + j + \bar{\jmath}$	$r = \bar{\jmath} - j$	$\hat{\mathcal{C}}_{R(j,\bar{\jmath})}$
$\mathcal{B}^1 \cap \bar{\mathcal{C}}_2$	$\Delta = 1 + \bar{\jmath} + 2R$	$r = \bar{\jmath} + 1$	$\mathcal{D}_{R(0,\bar{\jmath})}$
$\bar{\mathcal{B}}_2 \cap \mathcal{C}^1$	$\Delta = 1 + j + 2R$	$-r = j + 1$	$\bar{\mathcal{D}}_{R(j,0)}$
$\mathcal{B}^1 \cap \mathcal{B}^2 \cap \bar{\mathcal{C}}_2$	$\Delta = r = 1 + \bar{\jmath}$	$R = 0$	$\mathcal{D}_{0(0,\bar{\jmath})}$
$\mathcal{C}^1 \cap \bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$\Delta = -r = 1 + j$	$R = 0$	$\bar{\mathcal{D}}_{0(j,0)}$

Table 4.1: Shortening conditions for the unitary irreducible representations of the $\mathcal{N}=2$ superconformal algebra.

so-called chiral multiplets $\mathcal{E}_r$ which obey two $\mathcal{B}$-type shortening conditions and are associated with the physics of the Coulomb branch of $\mathcal{N}=2$ theories. Also prominent are 1/2 BPS multiplets, denoted by $\hat{\mathcal{B}}_R$, which obey two $\mathcal{B}$-type shortening conditions but of different chirality, these multiplets are associated with Higgs branch physics.

We will be mostly interested in the multiplet $\hat{\mathcal{C}}_{0(0,0)}$. Multiplets of the type $\hat{\mathcal{C}}_{R(j,\bar{\jmath})}$ obey semi-shortening conditions and the anti-commutation relation of the supercharges combine to give a generalized conservation equation. The special case $\hat{\mathcal{C}}_{0(0,0)}$ contains a spin two conserved current and we therefore identify it as the stress-tensor multiplet. In this work we will not consider theories that can be factorized as the product of two local theories, we will therefore assume a unique $\hat{\mathcal{C}}_{0(0,0)}$ multiplet. The multiplet also contains a conserved spin one operator which corresponds to the $SU(2)_R \times U(1)_r$ R-symmetry current.

Our goal is to study the super OPE of $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{C}}_{0(0,0)}$. In order to accomplish this we will carry out a detailed superspace analysis of three-point functions. Using $\mathcal{N}=2$ superspace language the stress-tensor multiplet can be represented by a superfield $\mathcal{J}$ that satisfies the conservation equation,

$$D^{\alpha i} D^j_\alpha \mathcal{J} = 0 \,, \qquad \bar{D}^i_{\dot{\alpha}} \bar{D}^{j\dot{\alpha}} \mathcal{J} = 0 \,, \tag{4.8}$$

where D^i_α and $\bar{D}_{\dot\alpha\, i}$ are $\mathcal{N}=2$ covariant derivatives and,

$$\mathcal{J}(x,\theta,\bar\theta) = J(x) + J^{i}_{\alpha\dot\alpha\, j}\theta^\alpha_i \bar\theta^{j\,\dot\alpha} + \dots \,. \tag{4.9}$$

Both the scalar $J(x)$ and current $J^{(ij)}_{\alpha\dot\alpha}(x)$ will be of particular importance to us.

$2d$ chiral algebra and analytic bound on c. As we will see below, the super OPE expansion of $\hat{\mathcal{C}}_{0(0,0)}$ will allow us to obtain an analytic bound on the central charge c. Of prime importance in this analysis will be the existence of a protected subsector of observables present in any $\mathcal{N}=2$ SCFT, whose correlation functions are described by a $2d$ chiral algebra. We will review this construction with some detail in section 3, for now let us just give a short outline of the calculation. Four-dimensional operators described by the chiral algebra sit in multiplets of the type,

$$\hat{\mathcal{B}}_R\,, \qquad \mathcal{D}_{R(0,\bar\jmath)}\,, \qquad \bar{\mathcal{D}}_{R(j,0)}\,, \qquad \hat{\mathcal{C}}_{0(j,\bar\jmath)}\,. \tag{4.10}$$

The $2d$ operator associated with the $\hat{\mathcal{C}}_{0(0,0)}$ multiplet is the $2d$ holomorphic stress-tensor, and it can be built using the $SU(2)_R$ current $J^{(ij)}_{\alpha\dot\alpha}(x)$,

$$J^{(ij)}_{\alpha\dot\alpha}(x) \to T(z)\,. \tag{4.11}$$

The $2d$ stress-tensor correlator constitutes a solvable truncation of the full four-point function of four currents $J^{(ij)}_{\alpha\dot\alpha}(x)$, and can be completely fixed by symmetry. This correlator can be expanded in conformal blocks associated with the multiplets listed in (4.10), and unitarity of the four-dimensional theory implies an analytic bound on c valid for any interacting $\mathcal{N}=2$ SCFT.

Crossing symmetry and numerical bounds on a/c. The supersymmetric selection rules are also relevant for the crossing symmetry equation of the superconformal primary

$J(x)$. This is a more challenging calculation and we will only present some partial results. The motivation for this is that using numerical bootstrap techniques we would then have access to the a-anomaly coefficient. This coefficient plays no role in the $2d$ chiral algebra, and cannot be bounded analytically, at least not with the techniques used in this paper. To obtain bounds on a one has to resort to numerics. In order to write crossing symmetry a fundamental ingredient is the conformal block expansion of the $\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle$ correlator. An attractive challenge for the $\mathcal{N} = 2$ bootstrap program would be to recover or even improve on the bounds found in [110] whose supersymmetric version reads,

$$\frac{1}{2} \leq \frac{a}{c} \leq \frac{5}{4}\,. \tag{4.12}$$

In section 5 the super OPE selection rules will help us understand how the different $\mathcal{N} = 2$ multiplets contribute to the J correlator, a necessary first step before a crossing symmetry equation can be written.

4.3 Three-point functions

We will now study all possible three point functions $\langle \mathcal{J}\mathcal{J}\mathcal{O}\rangle$ between two stress-tensor multiplets and a third arbitrary operator. The correlator for three stress-tensor multiplets $\langle \mathcal{J}\mathcal{J}\mathcal{J}\rangle$ was studied by Kuzenko and Theisen in [111]. We will use their notation and borrow some of their results. The starting point is the general expression for three-point functions in $\mathcal{N} = 2$ superspace [112, 113]

$$\langle \mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{O}^{\mathcal{I}}(z_3)\rangle = \frac{1}{(x_{\bar{1}3})^2(x_{\bar{3}1})^2(x_{\bar{2}3})^2(x_{\bar{3}2})^2} H^{\mathcal{I}}(\mathbf{Z}_3)\,, \tag{4.13}$$

where $\mathcal{I} = (\alpha, \dot\alpha, R, r)$ is a collective index that labels the irreducible representation to which $\mathcal{O}$ belongs. The (anti-)chiral combinations of coordinates are,

$$x_{\bar{1}2}^{\dot\alpha\alpha} = -x_{2\bar{1}}^{\dot\alpha\alpha} = x_{1-}^{\dot\alpha\alpha} - x_{2+}^{\dot\alpha\alpha} - 4\mathrm{i}\,\theta_{2\,i}^{\alpha}\bar\theta_{1}^{\dot\alpha i}\,, \tag{4.14}$$

$$\theta_{12} = \theta_1 - \theta_2\,, \qquad \bar\theta_{12} = \bar\theta_1 - \bar\theta_2\,, \tag{4.15}$$

with $x_{\pm}^{\dot\alpha\alpha} = x^{\dot\alpha\alpha} \mp 2\mathrm{i}\theta_i^\alpha\bar\theta^{\dot\alpha\,i}$. The argument of H is given by three superconformally covariant coordinates $\mathbf{Z}_3 = (\mathbf{X}_3, \Theta_3, \bar\Theta_3)$,

$$\mathbf{X}_{3\,\alpha\dot\alpha} = \frac{x_{3\bar{1}\,\alpha\dot\beta}\,x_{\bar{1}2}^{\dot\beta\beta}\,x_{2\bar{3}\,\beta\dot\alpha}}{(x_{3\bar{1}})^2(x_{2\bar{3}})^2}\,, \qquad \bar{\mathbf{X}}_{3\,\alpha\dot\alpha} = \mathbf{X}^\dagger_{3\,\alpha\dot\alpha} = -\frac{x_{3\bar{2}\,\alpha\dot\beta}\,x_{\bar{2}1}^{\dot\beta\beta}\,x_{1\bar{3}\,\beta\dot\alpha}}{(x_{3\bar{2}})^2(x_{1\bar{3}})^2}\,, \tag{4.16}$$

$$\Theta^i_{3\,\alpha} = \mathrm{i}\left(\frac{x_{\bar{2}3\,\alpha\dot\alpha}}{x^2_{\bar{2}3}}\bar\theta_{32}^{\dot\alpha i} - \frac{x_{\bar{1}3\,\alpha\dot\alpha}}{x^2_{\bar{1}3}}\bar\theta_{31}^{\dot\alpha i}\right)\,, \qquad \bar\Theta_{3\,\dot\alpha\,i} = \mathrm{i}\left(\theta_{32\,i}^{\alpha}\frac{x_{\bar{3}2\,\alpha\dot\alpha}}{x^2_{\bar{3}2}} - \theta_{31\,i}^{\alpha}\frac{x_{\bar{3}1\,\alpha\dot\alpha}}{x^2_{\bar{3}1}}\right)\,. \tag{4.17}$$

An important relation which will play a key role in our computations is

$$\bar{\mathbf{X}}_{3\,\alpha\dot\alpha} = \mathbf{X}_{3\,\alpha\dot\alpha} - 4\mathrm{i}\,\Theta^i_{3\,\alpha}\bar\Theta_{3\,\dot\alpha\,i}\,. \tag{4.18}$$

In addition, the function H satisfies the scaling condition,

$$H^{\mathcal{I}}(\lambda\bar\lambda\mathbf{X}_3, \lambda\Theta_3, \bar\lambda\bar\Theta_3) = \lambda^{2a}\bar\lambda^{2\bar{a}}H^{\mathcal{I}}(\mathbf{X}_3, \Theta_3, \bar\Theta_3)\,, \tag{4.19}$$

with $a - 2\bar{a} = 2 - q$ and $\bar{a} - 2a = 2 - \bar{q}$, where $\Delta = q + \bar{q}$ and $r = q - \bar{q}$. Extra restrictions are obtained by imposing the conservation equations of $\mathcal{J}$, these imply,

$$\frac{\partial^2}{\partial\Theta^i_{3\,\alpha}\partial\Theta_3^{\alpha\,j}}H^{\mathcal{I}}(\mathbf{Z}_3) = 0\,, \qquad \frac{\partial^2}{\partial\bar\Theta^{\dot\alpha}_{3\,i}\partial\bar\Theta_{3\,\dot\alpha\,j}}H^{\mathcal{I}}(\mathbf{Z}_3) = 0\,, \tag{4.20}$$

$$\mathcal{D}^\alpha_i\mathcal{D}_{\alpha\,j}\,H^{\mathcal{I}}(\mathbf{Z}_3) = 0\,, \qquad \tilde{\mathcal{D}}^{\dot\alpha\,i}\tilde{\mathcal{D}}^j_{\dot\alpha}\,H^{\mathcal{I}}(\mathbf{Z}_3) = 0\,, \tag{4.21}$$

where

$$\mathcal{D}_{\dot\alpha\,i} = \frac{\partial}{\partial\Theta_3^{\alpha\,i}} + 4\mathrm{i}\bar\Theta^{\dot\alpha}_{3\,i}\frac{\partial}{\partial\mathbf{X}_3^{\dot\alpha\alpha}}\,, \qquad \tilde{\mathcal{D}}^{\dot\alpha\,i} = \frac{\partial}{\partial\bar\Theta_{3\,\dot\alpha\,i}} - 4\mathrm{i}\Theta^i_{3\,\alpha}\frac{\partial}{\partial\mathbf{X}_{3\,\alpha\dot\alpha}}\,. \tag{4.22}$$

In order to see how these restrictions are imposed, let us work out an example in detail. For an operator $\mathcal{O}$ which is a scalar under Lorentz and $SU(2)_R \times U(1)$, equations (4.20) imply that H can be at most quadratic in Θ_3 and $\bar{\Theta}_3$. Thus, we will consider the following ansatz:

$$H(\mathbf{Z_3}) = f(\mathbf{X}_3) + g_{\alpha\dot{\alpha}}(\mathbf{X}_3)\Theta_3^{\alpha}\bar{\Theta}_3^{\dot{\alpha}} + h_{\alpha\beta\dot{\alpha}\dot{\beta}}(\mathbf{X}_3)\Theta_3^{\alpha\beta}\bar{\Theta}_3^{\dot{\alpha}\dot{\beta}}\,, \tag{4.23}$$

where

$$\Theta_3^{\alpha\beta} = \Theta_3^{\alpha i}\Theta_3^{\beta j}\,\varepsilon_{ij}\,, \qquad \bar{\Theta}_3^{\dot{\alpha}\dot{\beta}} = \bar{\Theta}_{3\,i}^{\dot{\alpha}}\Theta_{3\,j}^{\dot{\beta}}\,\varepsilon^{ij}\,. \tag{4.24}$$

This is the most general expression consistent with $SU(2)_R \times U(1)_r$ invariance quadratic in the Θs. Next, we impose the scaling condition (4.19),

$$H(\lambda\bar{\lambda}\mathbf{X}_3, \lambda\Theta_3, \bar{\lambda}\bar{\Theta}_3) = \lambda^{-4-\Delta/2}\bar{\lambda}^{-4-\Delta/2}H(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3)\,. \tag{4.25}$$

Hence, the functions f, g, and h are known up to an overall constant:

$$H(\mathbf{Z_3}) = a_1\frac{1}{(\mathbf{X}_3^2)^{2-\frac{\Delta}{2}}} + a_2\frac{\Theta_3^{\alpha}\mathbf{X}_{3\,\alpha\dot{\alpha}}\bar{\Theta}_3^{\dot{\alpha}}}{(\mathbf{X}_3^2)^{3-\frac{\Delta}{2}}} + a_3\frac{\Theta^{\alpha\beta}\mathbf{X}_{3\,\alpha\dot{\alpha}}\mathbf{X}_{3\,\beta\dot{\beta}}\bar{\Theta}^{\dot{\alpha}\dot{\beta}}}{(\mathbf{X}_3^2)^{4-\frac{\Delta}{2}}}\,. \tag{4.26}$$

Our correlator (4.13) should also be invariant under the exchange $z_1 \leftrightarrow z_2$ which implies $(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3) \to (-\bar{\mathbf{X}}_3, -\Theta_3, -\bar{\Theta}_3)$, as can be checked from (4.17). We will call this symmetry $\mathbb{Z}_2$ for short. Then,

$$H(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3) = H(-\bar{\mathbf{X}}_3, -\Theta_3, -\bar{\Theta}_3)\,. \tag{4.27}$$

This condition turns out to be very restrictive. In particular, if a function satisfies (4.20) and the $\mathbb{Z}_2$ condition, it also satisfies equations (4.21). Fixing the correlator is now a standard exercise in Grassmann algebra, we Taylor expand (4.27) in powers of the Grassmann variables and equate coefficients in both sides in order to fix (a_1, a_2, a_3). Details of our calculations along with some superspace identities are presented in appendix B.

For arbitrary Δ there is a unique solution given by

$$(a_1, a_2, a_3) = c_{\mathcal{J}\mathcal{J}\mathcal{O}}(1\,,\, \mathrm{i}(\Delta-4)\,,\, -\tfrac{1}{3}(\Delta-4)(\Delta-6))\,. \tag{4.28}$$

This solution is our generalization of the $\langle \mathcal{J}\mathcal{J}\mathcal{J}\rangle$ correlator for the case in which the third operator is a long multiplet $\mathcal{A}^{\Delta}_{0,0(0,0)}$ with unrestricted conformal dimension Δ.

For the special case $\Delta = 2$ the long multiplet hits its unitarity bound and splits according to,

$$\mathcal{A}^{2}_{0,0(0,0)} = \hat{\mathcal{C}}_{0(0,0)} + \mathcal{D}_{1(0,0)} + \bar{\mathcal{D}}_{1(0,0)} + \hat{\mathcal{B}}_2\,. \tag{4.29}$$

The results of this section imply that $\mathcal{D}$ and $\hat{\mathcal{B}}$ multiplets are not allowed.[1] Then, for $\Delta = 2$ the only surviving term in (4.29) is $\hat{\mathcal{C}}_{0(0,0)}$, and we just recover the $\langle \mathcal{J}\mathcal{J}\mathcal{J}\rangle$ correlator solution:

$$(a_1, a_2, a_3) = c^{(1)}_{\mathcal{J}\mathcal{J}\mathcal{J}}(1\,, -2\,\mathrm{i}\,, 0) + c^{(2)}_{\mathcal{J}\mathcal{J}\mathcal{J}}(0\,, 0\,, 1)\,. \tag{4.30}$$

That is, there are two independent structures. These two structures can be associated to the a and c anomaly coefficients, the exact relations were worked out in [111],

$$c^{(1)}_{\mathcal{J}\mathcal{J}\mathcal{J}} = \frac{3}{32\pi^6}(4\,a - c)\,, \qquad c^{(2)}_{\mathcal{J}\mathcal{J}\mathcal{J}} = \frac{1}{8\pi^6}(4\,a - 5\,c)\,. \tag{4.31}$$

The presence of two parameters is due to the fact that the last term in (4.26) is automatically symmetric under $z_1 \leftrightarrow z_2$ when $\Delta = 2$,[2]

$$\frac{\Theta^{\alpha\beta}\mathbf{X}_{\alpha\dot\alpha}\mathbf{X}_{\beta\dot\beta}\bar\Theta^{\dot\alpha\dot\beta}}{(\mathbf{X}^2)^3} = \frac{\Theta^{\alpha\beta}\bar{\mathbf{X}}_{\alpha\dot\alpha}\bar{\mathbf{X}}_{\beta\dot\beta}\bar\Theta^{\dot\alpha\dot\beta}}{(\bar{\mathbf{X}}^2)^3}\,. \tag{4.32}$$

Another way to phrase this, is that there is a "nilpotent invariant", namely, a purely fermionic term that satisfies all the symmetry requirements. It implies that we can not reconstruct the

[1] We refer the reader to (4.52) where we have collected in a single equation the super OPE selections rules obtained in this section.

[2] From now on we will ignore the subindex 3 in $(\mathbf{X}_3, \Theta_3, \bar\Theta_3)$.

full superspace three-point function starting from the three-point function of the superconformal primaries [112]. This is a generic property of superconformal field theories, unlike the pure conformal case in which three-point functions of descendants can always be obtained from that of primaries by taking derivatives. Although nilpotent invariants are to be expected, for some special cases it is impossible to build three-point invariants that satisfy all the symmetries of the correlator. Well known cases are 1/2 BPS operators in $\mathcal{N}=2$ and $\mathcal{N}=4$ theories [114, 115, 116, 117] and (anti)chiral operators in $\mathcal{N}=2$ and $\mathcal{N}=1$ theories [118, 101]. As we will see below, nilpotent invariants will also be present when we consider operators with spin.

$\mathcal{N}=1$ **check**: As a check on our result, let us reduce it to $\mathcal{N}=1$ superspace language and compared it the known solutions of [102, 119, 120]. Using the coefficients (4.28) in (4.23) and rewriting in denominators in terms of $\mathbf{X}\cdot\bar{\mathbf{X}}$. Setting the $i=2$ components to zero we obtain,

$$\Theta_{\alpha\, i=1} \to \Theta_\alpha\,, \qquad \Theta_{\alpha\, i=2} \to 0\,, \tag{4.33}$$

where Θ_α is the analogous $\mathcal{N}=1$ coordinate. Our solution reduces to,

$$H(\mathbf{Z}) = \frac{1}{(\mathbf{X}\cdot\bar{\mathbf{X}})^{2-\frac{\Delta}{2}}}\left(1-\frac{1}{4}(\Delta-4)(\Delta-6)\frac{\Theta^2\bar{\Theta}^2}{(\bar{\mathbf{X}}\cdot\mathbf{X})}\right)\,, \tag{4.34}$$

in perfect agreement with the $\mathcal{N}=1$ result of [120, 119, 102].

The procedure is now clear:

- Write the most general ansatz consistent with (4.20).
- Fix the $\mathbf{X}$-dependence using the scaling condition (4.19)
- Fix the arbitrary coefficients by imposing the $\mathbb{Z}_2$ symmetry (4.27).

We now apply this strategy to all possible combinations of Lorentz and $SU(2)_R \times U(1)_r$ quantum numbers in order to find the $\mathcal{N}=2$ selection rules for the OPE of two stress-tensor multiplets.

4.3.1 Solutions

$\mathcal{A}^{\Delta}_{0,0(\frac{\ell}{2},\frac{\ell}{2})}$

The most general ansatz for arbitrary ℓ consistent with the conditions discussed above is

$$\begin{aligned} H(\mathbf{Z}) = & \frac{\mathbf{X}_{\alpha_1\dot\alpha_1}\dots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}}{(\mathbf{X}^2)^{2-\frac{\Delta-\ell}{2}}}\left(a_1 + a_2\,\frac{\Theta^{\alpha i}\mathbf{X}_{\alpha\dot\alpha}\bar\Theta^{\dot\alpha}_i}{\mathbf{X}^2} + a_3\,\frac{\Theta^{\alpha\beta}\mathbf{X}_{\alpha\dot\alpha}\mathbf{X}_{\beta\dot\beta}\bar\Theta^{\dot\alpha\dot\beta}}{(\mathbf{X}^2)^2}\right) \\ & + \frac{\mathbf{X}_{\alpha_2\dot\alpha_2}\dots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}}{(\mathbf{X}^2)^{2-\frac{\Delta-\ell}{2}}}\left(a_4\,\Theta^i_{\alpha_1}\bar\Theta_{\dot\alpha_1\,i} + a_5\,\frac{\bar\Theta_{\dot\alpha_1\dot\beta}\mathbf{X}^{\dot\beta\beta}\Theta_{\beta\alpha_1}}{\mathbf{X}^2}\right) \\ & + a_6\frac{\mathbf{X}_{\alpha_3\dot\alpha_3}\dots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}}{(\mathbf{X}^2)^{2-\frac{\Delta-\ell}{2}}}\Theta_{\alpha_1\alpha_2}\bar\Theta_{\dot\alpha_1\dot\alpha_2}\,, \end{aligned} \tag{4.35}$$

where it is understood that the indices $(\alpha_1,\dots,\alpha_\ell)$ and $(\dot\alpha_1,\dots,\dot\alpha_\ell)$ are symmetrized with weight one. Imposing the $\mathbb{Z}_2$ symmetry we find, for the odd case,

$$\vec a = c_{\mathcal{J}\mathcal{J}\mathcal{O}}\left(0,\frac{1}{2(\Delta-\ell)},\frac{\mathrm{i}(\Delta-6-\ell)}{4(\Delta-2)},\frac{1}{\Delta-4-\ell},\frac{\mathrm{i}(\Delta-2-\ell)}{2(\Delta-2)},\frac{\mathrm{i}(1-\ell)}{(\Delta-4-\ell)}\right). \tag{4.36}$$

For $\ell=1$ the last structure in (4.35) can not contribute.

For the ℓ even case we find two different solutions

$$\begin{aligned} \vec a = & c^{(1)}_{\mathcal{J}\mathcal{J}\mathcal{O}}\left(0,0,\frac{1}{2}(\Delta-6-\ell),(3\Delta+\ell-6),0,\frac{(3(\Delta-2)^2-2\ell-\ell^2)}{(\Delta-4-\ell)}\right) \\ & + c^{(2)}_{\mathcal{J}\mathcal{J}\mathcal{O}}\left(\mathrm{i}\frac{1}{2\ell},-\frac{(\Delta-4-\ell)}{2\ell},-\mathrm{i}\frac{(\Delta+\ell-2)(\Delta-4-\ell)(\Delta-6-\ell)}{2\ell(3\Delta+\ell-6)},\right. \\ & \left. 1,0,-2\mathrm{i}\frac{(\Delta-3)(\Delta-2+\ell)}{(3\Delta+\ell-6)}\right) \end{aligned} \tag{4.37}$$

The two-parameter solution is due to the existence of three-point "nilpotent invariant" that can only be constructed when the spin is even. Indeed, the object

$$\begin{aligned} & \frac{1}{2}(\Delta-6-\ell)\frac{\mathbf{X}_{\alpha_1\dot\alpha_1}\dots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}}{(\mathbf{X}^2)^{4-\frac{\Delta-\ell}{2}}}\,\Theta^{\alpha\beta}\mathbf{X}_{\alpha\dot\alpha}\mathbf{X}_{\beta\dot\beta}\bar\Theta^{\dot\alpha\dot\beta} + (3\Delta+\ell-6)\frac{\mathbf{X}_{\alpha_2\dot\alpha_2}\dots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}}{(\mathbf{X}^2)^{2-\frac{\Delta-\ell}{2}}}\,\Theta^i_{\alpha_1}\bar\Theta_{\dot\alpha_1\,i} \\ & + \frac{(3(\Delta-2)^2-2\ell-\ell^2)}{(\Delta-4-\ell)}\frac{\mathbf{X}_{\alpha_3\dot\alpha_3}\dots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}}{(\mathbf{X}^2)^{2-\frac{\Delta-\ell}{2}}}\Theta_{\alpha_1\alpha_2}\bar\Theta_{\dot\alpha_1\dot\alpha_2}\,, \end{aligned} \tag{4.38}$$

satisfies all the constraints imposed by $\mathcal{N} = 2$ superconformal symmetry. As a consequence, the superconformal block will have an undetermined parameter. In [102] superconformal blocks for general scalar operators were obtained where the same happens, the block has a number of free parameters, in that case things can be improved if one imposes conservation or chirality conditions. Our result implies that in $\mathcal{N} = 2$ theories, even imposing the conservation condition is not enough, and there will be an unfixed parameter in the superconformal block expression. It would be interesting to understand whether this parameter has some physical meaning, like in the $\Delta = 2$, $\ell = 0$ case, where they are identified with anomaly coefficients.

At the unitarity bound we have the splitting,

$$\mathcal{A}^{2+\ell}_{0,0(\frac{\ell}{2},\frac{\ell}{2})} = \hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})} + \hat{\mathcal{C}}_{\frac{1}{2}(\frac{\ell-1}{2},\frac{\ell}{2})} + \hat{\mathcal{C}}_{\frac{1}{2}(\frac{\ell}{2},\frac{\ell-1}{2})} + \hat{\mathcal{C}}_{1(\frac{\ell-1}{2},\frac{\ell-1}{2})} \,. \tag{4.39}$$

The multiplets $\hat{\mathcal{C}}_{\frac{1}{2}(\frac{\ell-1}{2},\frac{\ell}{2})}$ and $\hat{\mathcal{C}}_{\frac{1}{2}(\frac{\ell}{2},\frac{\ell-1}{2})}$ are not allowed by the selection rules (see (4.52)). The $\hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})}$ multiplets contain higher spin currents and are not expected to appear in interacting theories, with the exception of $\ell = 0$.

$\mathcal{A}^{\Delta}_{0,0(\frac{\ell+2}{2},\frac{\ell}{2})}$

We also found solutions for complex long multiplets,

$$\begin{aligned} H(\mathbf{Z}) = \frac{\mathbf{X}_{\alpha_1\dot{\alpha}_1}\ldots\mathbf{X}_{\alpha_\ell\dot{\alpha}_\ell}}{(\mathbf{X}^2)^{3-\frac{\Delta-\ell}{2}}} \left(a_1\, \Theta^i_{\alpha_{\ell+1}} \mathbf{X}_{\alpha_{\ell+2}\,\dot{\alpha}} \bar{\Theta}^{\dot{\alpha}}_i + a_2\, \epsilon_{\alpha_{\ell+1}\,\alpha} \mathbf{X}_{\alpha_{\ell+2}\,\dot{\alpha}} \frac{\Theta^{\alpha\beta}\mathbf{X}_{\beta\dot{\beta}}\bar{\Theta}^{\dot{\beta}\dot{\alpha}}}{\mathbf{X}^2} \right) \\ + a_3 \frac{\mathbf{X}_{\alpha_1\dot{\alpha}_1}\ldots\mathbf{X}_{\alpha_{\ell-1}\,\dot{\alpha}_{\ell-1}}}{(\mathbf{X}^2)^{3-\frac{\Delta-\ell}{2}}} \Theta_{\alpha_\ell\,\alpha_{\ell+1}} \mathbf{X}_{\alpha_{\ell+2}\,\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\Theta}_{\dot{\alpha}_\ell\,\dot{\beta}} \,. \end{aligned} \tag{4.40}$$

For ℓ even we have $(a_1, a_2, a_3) = c_{\mathcal{J}\mathcal{J}\mathcal{O}}\,(0, (\Delta-6-\ell), 2(\Delta-2))$, while for ℓ odd $(a_1, a_2, a_3) = c_{\mathcal{J}\mathcal{J}\mathcal{O}}\,(2, \mathrm{i}(\Delta-6-\ell), -2\mathrm{i}\ell)$. For $\ell = 0$ there is no solution.

At the unitarity bound we have,

$$\mathcal{A}^{3+\ell}_{0,0(\frac{\ell+2}{2},\frac{\ell}{2})} = \mathcal{C}_{0,0(\frac{\ell+2}{2},\frac{\ell}{2})} + \mathcal{C}_{\frac{1}{2},\frac{1}{2}(\frac{\ell+1}{2},\frac{\ell}{2})} \,. \tag{4.41}$$

The multiplet $\mathcal{C}_{\frac{1}{2},\frac{1}{2}(\frac{\ell+1}{2},\frac{\ell}{2})}$ is not allowed by the selection rules (see (4.52)).

$\mathcal{A}^{\Delta}_{0,0(\frac{\ell+4}{2},\frac{\ell}{2})}$

Finally, there is another long multiplet

$$H(\mathbf{Z}) = \frac{c_{\mathcal{J}\mathcal{J}\mathcal{O}}}{(\mathbf{X}^2)^{4-\frac{\Delta-\ell}{2}}} \mathbf{X}_{\alpha_1\dot{\alpha}_1} \ldots \mathbf{X}_{\alpha_\ell\dot{\alpha}_\ell} \mathbf{X}_{\alpha_{\ell+1}\dot{\alpha}} \mathbf{X}_{\alpha_{\ell+2}\dot{\beta}} \Theta_{\alpha_{\ell+3}\alpha_{\ell+4}} \bar{\Theta}^{\dot{\alpha}\dot{\beta}} \,, \tag{4.42}$$

with $c_{\mathcal{J}\mathcal{J}\mathcal{O}} \neq 0$ only for ℓ even.

At the unitarity bound we have,

$$\mathcal{A}^{4+\ell}_{0,0(\frac{\ell+4}{2},\frac{\ell}{2})} = \mathcal{C}_{0,0(\frac{\ell+4}{2},\frac{\ell}{2})} + \mathcal{C}_{\frac{1}{2},\frac{1}{2}(\frac{\ell+3}{2},\frac{\ell}{2})} \,. \tag{4.43}$$

The multiplet $\mathcal{C}_{\frac{1}{2},\frac{1}{2}(\frac{\ell+3}{2},\frac{\ell}{2})}$ is not allowed by the selection rules (see (4.52)).

$\bar{\mathcal{C}}_{0,-3(\frac{\ell+2}{2},\frac{\ell}{2})}$

We also found solutions that fix the conformal dimension Δ,

$$H(\mathbf{Z}) = \frac{c_{\mathcal{J}\mathcal{J}\mathcal{O}}}{(\mathbf{X}^2)^{\frac{3}{2}-\frac{\Delta-\ell}{2}}} \mathbf{X}_{\alpha_1\dot{\alpha}_1} \cdots \mathbf{X}_{\alpha_\ell\dot{\alpha}_\ell} \Theta_{\alpha_{\ell+1}\alpha_{\ell+2}} \,, \tag{4.44}$$

has nonzero a for $\Delta = 5 + \ell$, and $\ell \geq 0$ even. This is precisely the unitarity bound for this quantum numbers and corresponds to a semi-short multiplet of the $\mathcal{C}$-type.

$\bar{\mathcal{C}}_{\frac{1}{2},-\frac{3}{2}(\frac{\ell+1}{2},\frac{\ell}{2})}$

We can also have multiplets that transform non-trivially under $SU(2)_R$ representations,

$$H(\mathbf{Z})=\frac{\mathbf{X}_{\alpha_1\dot{\alpha}_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot{\alpha}_{\ell-1}}}{(\mathbf{X}^2)^{\frac{9}{4}-\frac{\Delta-\ell}{2}}}\left(a_1\mathbf{X}_{\alpha_\ell\dot{\alpha}_\ell}\Theta^i_{\alpha_{\ell+1}}+a_2\frac{\mathbf{X}_{\alpha_\ell\dot{\alpha}_\ell}\mathbf{X}^{\dot{\beta}\beta}}{\mathbf{X}^2}\Theta_{\alpha_{\ell+1}\beta}\bar{\Theta}^i_{\dot{\beta}}+a_3\Theta_{\alpha_\ell\alpha_{\ell+1}}\bar{\Theta}^i_{\dot{\alpha}_\ell}\right). \tag{4.45}$$

H is nonvanishing only for $\Delta=\frac{9}{2}+\ell$ which is the unitarity bound for these quantum numbers. The solution is $(a_1,a_2,a_3)=c_{\mathcal{J}\mathcal{J}\mathcal{O}}\,(1,0,\mathrm{i}\,\ell)$ for ℓ odd and $(a_1,a_2,a_3)=c_{\mathcal{J}\mathcal{J}\mathcal{O}}\,(0,1,0)$ for ℓ even.

$\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}$

For $SU(2)_R$ triplets we find the following family,

$$H=\frac{c_{\mathcal{J}\mathcal{J}\mathcal{O}}}{(\mathbf{X}^2)^{2-\frac{\Delta-\ell}{2}}}\mathbf{X}_{\alpha_1\dot{\alpha}_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot{\alpha}_{\ell-1}}\Theta^{(i}_{\alpha_\ell}\bar{\Theta}^{j)}_{\dot{\alpha}_\ell} \tag{4.46}$$

This structure is nonvanishing only for $\Delta=4+\ell$, which is again the unitarity bound. Now, the flavor current sits in $\hat{\mathcal{B}}_1$ multiplet which is a triplet under $SU(2)_R$ and has $\ell=0$. Its superspace field was denoted by L^{ij} in [111] and it was found that $\langle\mathcal{J}\mathcal{J}L^{ij}\rangle=0$. Our solution is consistent with their result.

For $SU(2)_R$ representations higher than $R=1$ no solutions exist due to the condition that the correlator be at most quadratic in Θ and $\bar{\Theta}$.

4.3.2 Extra solutions

In addition to the multiplets described above we found extra solutions.

Non-unitary $(R,r,j,\bar{\jmath}) = \left(\frac{1}{2}, -\frac{3}{2}, \frac{\ell}{2}, \frac{\ell+1}{2}\right)$ **solution**

The following structure is also allowed,

$$\begin{aligned} H =& \frac{\mathbf{X}_{\alpha_1\dot\alpha_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot\alpha_{\ell-1}}}{(\mathbf{X}^2)^{\frac{11}{4}-\frac{\Delta-\ell}{2}}}\Bigg(a_1\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}\mathbf{X}_{\alpha\dot\alpha_{\ell+1}}\Theta^{\alpha\, i} + a_2\frac{\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}\mathbf{X}_{\alpha\dot\alpha_{\ell+1}}\mathbf{X}_{\beta\dot\beta}\Theta^{\alpha\beta}\bar\Theta^{\dot\beta\, i}}{\mathbf{X}^2} \\ & + a_3\mathbf{X}_{\alpha\dot\alpha_\ell}\Theta^{\alpha}{}_{\alpha_\ell}\bar\Theta^{i}_{\dot\alpha_{\ell+1}}\Bigg) \end{aligned} \tag{4.47}$$

H is nonvanishing only for $\Delta = \frac{3}{2} - \ell$ for $(a_1,a_2,a_3) = c_{\mathcal{JJO}}(1, 2\,\mathrm{i}, 0)$ for $\ell = 0$, $(a_1,a_2,a_3) = c_{\mathcal{JJO}}(0, \ell+2, \ell)$ for ℓ odd, and $(a_1,a_2,a_3) = c_{\mathcal{JJO}}(1, \mathrm{i}\,(\ell+2), \mathrm{i}\,\ell)$ for ℓ even. This solution is below the unitarity bound and therefore of no interest to us. Similar non-unitary solutions were found in [119].

Non-unitary $(R,r,j,\bar{\jmath}) = \left(\frac{1}{2}, -\frac{3}{2}, \frac{\ell}{2}, \frac{\ell+3}{2}\right)$ **solution**

We also found

$$H(\mathbf{Z}) = c_{\mathcal{JJO}}\frac{\mathbf{X}_{\alpha_1\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}}{(\mathbf{X}^2)^{\frac{19}{4}-\frac{\Delta-\ell}{2}}}\mathbf{X}_{\alpha\dot\alpha_{\ell+1}}X_{\beta\dot\alpha_{\ell+2}}\Theta^{\alpha\beta}\bar\Theta^{i}_{\dot\alpha_{\ell+3}}\,. \tag{4.48}$$

H is nonvanishing only for $\Delta = \frac{3}{2} - \ell$ and only for $\ell \geq 1$ odd. As the case above, this is below the unitarity bound and has no relevance for this work.

$\mathcal{A}^{\frac{13}{2}+\ell}_{\frac{1}{2},-\frac{3}{2}(\frac{\ell+3}{2},\frac{\ell}{2})}$

Finally, we found a strange solution that corresponds to a long multiplet with fixed conformal dimension:

$$H(\mathbf{Z}) = \frac{c_{\mathcal{JJO}}}{(\mathbf{X}^2)^{\frac{13}{4}-\frac{\Delta-\ell}{2}}}\mathbf{X}_{\alpha_1\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}\mathbf{X}_{\alpha_{\ell+1}\dot\alpha}\Theta_{\alpha_{\ell+2}\alpha_{\ell+3}}\bar\Theta^{\dot\alpha\, i}\,. \tag{4.49}$$

The only restriction for this long multiplet is that the conformal dimension be above the $\mathcal{N}=2$ unitarity bound $\Delta = \frac{9}{2} + \ell$, it is then puzzling that our solution fixes its dimension to $\Delta = \frac{13}{2} + \ell$. Because it sits above the unitarity bound we can not interpret it as a contribution from a short multiplet. One possible explanation is that this multiplet corresponds to a theory that has enhanced $\mathcal{N}=4$ symmetry. $\mathcal{N}=2$ long multiplets with fixed conformal dimension appear if one decomposes $\mathcal{N} = 4$ multiplets. The OPE of two $\mathcal{N} = 4$ stress-tensors is well known [115, 116],

$$\begin{aligned} \mathcal{B}_{[0,2,0]} \times \mathcal{B}_{[0,2,0]} \sim \mathcal{B}_{[0,2,0]} + \mathcal{B}_{[0,4,0]} + \mathcal{B}_{[1,0,1]} + \mathcal{B}_{[1,2,1]} + \mathcal{B}_{[2,0,2]} \\ + \mathcal{C}_{[0,0,0],\ell} + \mathcal{C}_{[1,0,1],\ell} + \mathcal{C}_{[0,2,0],\ell} + \ldots \,, \end{aligned} \tag{4.50}$$

where the ... stand for long multiplets with unrestricted conformal dimension. In the decomposition of the $\mathcal{N} = 4$ stress-tensor multiplet we find, among other things, the $\mathcal{N} = 2$ stress-tensor multiplet,

$$\mathcal{B}_{[0,2,0]} = \ldots + \hat{\mathcal{C}}_{0(0,0)} + \ldots \,. \tag{4.51}$$

Our curious multiplet could appear in the decomposition of one of the multiplets in the RHS of (4.50). The $\mathcal{B}_{[0,p,0]}$ decompositions were worked out in [109] and our multiplet does not appear there, our guess is that is hiding somewhere in the $\mathcal{C}$ multiplets. In principle one could use the character techniques of [121] to confirm this suspicion, although straightforward, this type of calculation can still become quite involved. In the remainder, we will ignore this solution considering it an accident with no relevance to $\mathcal{N} = 2$ dynamics.

4.4 2d chiral algebra and central charge bound

The superspace analysis of the previous section allows us to write the super OPE selection rules for the $\mathcal{N} = 2$ stress-tensor multiplet,[3]

$$\begin{aligned} \hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{C}}_{0(0,0)} \sim \mathcal{I} + \hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})} + \hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})} + \mathcal{C}_{\frac{1}{2},\frac{3}{2}(\frac{\ell}{2},\frac{\ell+1}{2})} \\ + \mathcal{C}_{0,3(\frac{\ell}{2},\frac{\ell+2}{2})} + \mathcal{C}_{0,0(\frac{\ell+2}{2},\frac{\ell}{2})} + \mathcal{C}_{0,0(\frac{\ell+4}{2},\frac{\ell}{2})} \\ + \mathcal{A}^{\Delta}_{0,0(\frac{\ell}{2},\frac{\ell}{2})} + \mathcal{A}^{\Delta}_{0,0(\frac{\ell+2}{2},\frac{\ell}{2})} + \mathcal{A}^{\Delta}_{0,0(\frac{\ell+4}{2},\frac{\ell}{2})} \,. \end{aligned} \tag{4.52}$$

[3] To avoid cluttering we do not write the conjugate multiplets.

We will now use this information to obtain an analytic bound on the central charge c valid for any $\mathcal{N}=2$ superconformal field theory. To accomplish this, we will rely on the observation that any $\mathcal{N}=2$ SCFTs contains a closed subsector of operators isomorphic to a two-dimensional chiral algebra. Let us then start reviewing how chiral algebras appear in $\mathcal{N}=2$ SCFTs, for more details we refer the reader to the original paper [92] (see also [122, 123, 124, 125]).

It is possible to define a map that associates to any $\mathcal{N}=2$ SCFTs a two-dimensional chiral algebra:

$$4d \text{ SCFT} \qquad \rightarrow \qquad 2d \text{ Chiral Algebra}$$

whose correlation functions describe a protected subsector of the original four-dimensional theory. The construction of the two dimensional chiral algebra is obtained by going to the cohomology of a certain nilpotent supercharge

$$\mathbb{Q} = \mathcal{Q}^1_- + \bar{\mathcal{S}}^{2\dot{-}}\,, \tag{4.53}$$

where $\mathcal{Q}^i_\alpha$ and $\bar{\mathcal{S}}^{i\,\dot{\alpha}}$ are the standard supercharges of the $\mathcal{N}=2$ superconformal algebra. Fixing a plane $\mathbb{R}^2 \in \mathbb{R}^4$ and defining complex coordinates $(z,\bar{z})$ on it, the conformal symmetry restricted to the plane acts as $SL(2)\times\overline{SL(2)}$. The supercharge $\mathbb{Q}$ can be used to define holomorphic translations that are $\mathbb{Q}$-closed and anti-holomorphic translations that are $\mathbb{Q}$-exact:

$$[\mathbb{Q}, SL(2)] = 0\,, \qquad \{\mathbb{Q}, \text{something}\} = \widehat{SL(2)}\,, \tag{4.54}$$

where $\widehat{SL(2)} = \text{diag}\left(\overline{SL(2)}\times SL(2)_R\right)$ and $SL(2)_R$ is the complexification of the compact $SU(2)_R$ R-symmetry. Operators that belong to the cohomology of $\mathbb{Q}$ transform in chiral representations of the $SL(2)\times\widehat{SL(2)}$ subalgebra. This implies that they have meromorphic OPEs (module $\mathbb{Q}$-exact terms) and their correlation functions are meromorphic functions of their positions when restricted to the plane.

In order to identify the cohomology of $\mathbb{Q}$ we will consider operators at the origin, and then we will translate them across the plane using the $SL(2) \times \widehat{SL(2)}$ generators. As shown in [92], a necessary and sufficient condition for an operator to be in the cohomology of $\mathbb{Q}$ is,

$$\frac{1}{2}(\Delta - (j + \bar{\jmath})) - R = 0\,, \qquad r + (j - \bar{\jmath}) = 0\,. \tag{4.55}$$

We call this operators *Schur operators*, because they contribute to the Schur limit of the superconformal index [126]. It can be shown that Schur operators occupy the highest weight of their respective $SU(2)_R$ and Lorentz representations,

$$\mathcal{O}^{1\ldots1}_{+\ldots+\dot{+}\ldots\dot{+}}(0)\,. \tag{4.56}$$

Having identified the operator at the origin, we proceed to translate it using the $SL(2)\times\widehat{SL(2)}$ generators. Equation (4.54) implies that the anti-holomorphic dependence gets entangled with the $SU(2)_R$ structure due to the twisted nature of the $\widehat{SL(2)}$ generators. The coordinate dependence after translation is,

$$\mathcal{O}(z, \bar{z}) = u_{i_1}(\bar{z}) \ldots u_{i_k}(\bar{z}) \mathcal{O}^{(i_1 \ldots i_k)}(z, \bar{z}) \qquad \text{where} \qquad u_i(\bar{z}) = (1, \bar{z})\,. \tag{4.57}$$

By construction, these operators define cohomology classes with meromorphic correlators. For each cohomology class we define,

$$\mathcal{O}(z) = [\mathcal{O}(z, \bar{z})]_{\mathbb{Q}}\,. \tag{4.58}$$

That is, to any $4d$ Schur operator there is an associated $2d$ dimensional holomorphic operator. Schur operators have protected conformal dimension and therefore sit in shortened multiplets of the superconformal algebra. In table 4.2 we present the list of multiplets that contain a Schur operator and the holomorphic dimension h of the corresponding two-dimensional operator.

Multiplet	$\mathcal{O}_{\text{Schur}}$	h	r
$\hat{\mathcal{B}}_R$	$\Psi^{11\dots1}$	R	0
$\mathcal{D}_{R(0,\bar{\jmath})}$	$\bar{\mathcal{Q}}^1_{\dot{+}}\,\Psi^{11\dots1}_{\dot{+}\dots\dot{+}}$	$R+\bar{\jmath}+1$	$\bar{\jmath}+\frac{1}{2}$
$\bar{\mathcal{D}}_{R(j,0)}$	$\mathcal{Q}^1_{+}\,\Psi^{11\dots1}_{+\dots+}$	$R+j+1$	$-j-\frac{1}{2}$
$\hat{\mathcal{C}}_{R(j,\bar{\jmath})}$	$\mathcal{Q}^1_{+}\,\bar{\mathcal{Q}}^1_{\dot{+}}\,\Psi^{11\dots1}_{+\dots+\dot{+}\dots\dot{+}}$	$R+j+\bar{\jmath}+2$	$\bar{\jmath}-j$

Table 4.2: Four-dimensional superconformal multiplets that contain Schur operators, we denote the superconformal by Ψ. The second column indicates where in the multiplet the Schur operator sits. The third and fourth column give the two-dimensional quantum numbers in terms of $(R, j, \bar{\jmath})$.

4.4.1 Enhanced Virasoro symmetry

Among the list of multiplets in table 4.2 is the stress-tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$ and its Schur operator is the $SU(2)_R$ conserved current $J^{11}_{+\dot{+}}$. Its corresponding holomorphic operator is defined as $T(z) = [J_{+\dot{+}}(z,\bar{z})]_{\mathbb{Q}}$, and the four-dimensional $J_{+\dot{+}}(x)J_{+\dot{+}}(0)$ OPE implies,

$$T(z)T(0) \sim -\frac{6\,c_{4d}}{z^4} + 2\frac{T(0)}{z^2} + \frac{\partial T(0)}{z} + \dots . \tag{4.59}$$

We can therefore identify $T(z)$ as the $2d$ stress-tensor. The $2d$ central charge is,

$$c_{2d} = -12\,c_{4d}\,. \tag{4.60}$$

Unitarity of the four-dimensional theory implies that the two-dimensional theory is non-unitary. The holomorphic correlator of the stress-tensor can be completely fixed in terms of the central charge, and its relation to the parent theory in four dimensions will allow us to obtain an analytic bound on c. The holomorphic correlator of the stress-tensor is,

$$g(z) = 1 + z^4 + \frac{z^4}{(1-z)^4} + \frac{8}{c_{2d}}\left(z^2 + z^3 + \frac{z^4}{(1-z)^2} + \frac{z^4}{1-z}\right), \tag{4.61}$$

and admits the following expansion in $SL(2)$ blocks,

$$g(z)=\sum_{\ell=0}^{\infty} a_\ell \, z^\ell \, {}_2F_1(\ell,\ell,2\ell,z) \qquad \ell \quad \text{even}, \tag{4.62}$$

where ${}_2F_1$ is the standard hypergeometric function. Thanks to the $4d/2d$ correspondence we can interpret the $SL(2)$ blocks as contributions from four-dimensional multiplets containing Schur operators. Looking at the super OPE selection rules in (4.52) there are only two possible choices,

$$\hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})} \qquad \text{and} \qquad \hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})} \,. \tag{4.63}$$

The $\hat{\mathcal{C}}_{0(\frac{\ell}{2},\frac{\ell}{2})}$ multiplets contain higher spin currents and we do not expect them in an interacting theory [127, 128]. The only candidate then is $\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}$, the exact proportionality constant α between the OPE coefficients $\lambda^2_{\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}}$ and the $SL(2)$ coefficients a_ℓ can be carefully worked out, but we will not need it. The explicit expansion of (4.61) in terms of $SL(2)$ blocks was worked out in [129], in particular,

$$\lambda^2_{\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}} = \alpha\left(2-\frac{11}{15c_{4d}}\right) . \tag{4.64}$$

Unitarity of the four dimensional theory implies $\lambda^2_{\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}} \geq 0$ then,[4]

$$c_{4d} \geq \frac{11}{30} \, . \tag{4.65}$$

Let us note that in order to obtain this bound we only assumed $\mathcal{N}=2$ superconformal symmetry, existence of a stress-tensor, and absence of higher spin currents. Bounds of this type were obtained in [92] using the $\hat{\mathcal{B}}_1$ four-point function, in that case however, it is necessary to assume the existence of flavor symmetries whose conserved currents sit in $\hat{\mathcal{B}}_1$ multiplets. In the present case, our assumptions are weaker. A similar bound was also

[4]Because we have not calculated the exact proportionality constant, one could complain that an overall minus sign will invalidate our bound. However, common sense dictates that the sign should be positive, otherwise we will rule out every known interacting $\mathcal{N}=2$ SCFT.

obtained for $\mathcal{N}=4$ theories in [130], where absence of higher spin currents imply $c \geq \frac{3}{4}$.

Going through the $\mathcal{N}=2$ literature one can check that the simplest rank one Argyres-Douglas fixed point (sometimes denoted as H_0 due to its construction in F-theory) has central charge $c=\frac{11}{30}$ [37, 38, 34, 131], which precisely saturates our bound. The analytic bounds of [92] turned out to have interesting consequences for four-dimensional physics: the saturation of a bound was identified as a relation in the Higgs branch chiral ring due to the decoupling of the associated multiplet. It would be interesting to explore whether the absence of the $\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}$ multiplet is associated with some intrinsic structure that characterizes the H_0 theory.

From the two-dimensional point of view, the $2d$ chiral algebra that describes the H_0 theory has been conjectured to be the Yang-Lee minimal model [132]. Indeed, the $2d$ value of the central charge is $c_{2d}=-\frac{22}{5}$. Saturation of the bound implies the absence of the $\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}$ multiplet, from table 4.2 the associated $2d$ operator has holomorphic dimension 4. Hence, absence of $\hat{\mathcal{C}}_{1(\frac{1}{2},\frac{1}{2})}$ translate to the existence of a null state of dimension 4. Remarkably, one of the hallmarks of the Yang-Lee minimal a model is a level 4 null descendant of the identity, $\left(L_{-2}^2-\frac{3}{5}L_{-4}\right)|0\rangle$. Our results are then consistent with the conjectured correspondence. The Schur index of Argyres-Douglas fixed points and its relation to $2d$ chiral algebras was recently studied in [133, 134].

The vanishing of certain OPE coefficients has also been instrumental in characterizing the $3d$ critical Ising model using numerical bootstrap techniques [13, 15, 16]. One can then label the rank one H_0 theory as the “Ising model” of $\mathcal{N}=2$ superconformal theories, in the sense that it shares two of its most prominent features: minimum value of the central charge, and vanishing of certain OPE coefficients. Both features indicate that this superconformal fixed point sits in a very special place in the parameter space of $\mathcal{N}=2$ theories and a numerical treatment seems feasible [135].

4.5 Superconformal block analysis

The super selection rules are a necessary first step toward writing the conformal block expansion of the J correlator. The results of section 3 give a clearer picture of how this expansion works in the case of $\mathcal{N} = 2$ theories. Superconformal block expansions for 1/2 BPS and chiral operators for several combinations of supersymmetry and spacetime dimension have been worked out [118, 101, 115, 116, 117, 136]. There has been more success studying chiral and 1/2 BPS operators because one can construct superspaces in which they are naturally defined, and the analysis simplifies. Semi-short and long multiplets in general are harder to study, the work of [102] and [107] attempts to tackle the more general cases.

4.5.1 Quick review of conformal blocks

Given the four-point function of a scalar J one can use the OPE in order to write the four-point correlator as a sum of conformal blocks (also called conformal partial waves),

$$J(x)J(0) = \sum_{\mathcal{O}\in J\times J} \lambda_{JJ\mathcal{O}} C_{\Delta,\ell}(x,\mathcal{P})\mathcal{O}_{\Delta,\ell}(0)\,. \tag{4.66}$$

Plugging the OPE into the four-function,

$$\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle = \frac{1}{x_{12}^4 x_{34}^4} \sum_{\mathcal{O}\in J\times J} \lambda_{JJ\mathcal{O}}^2\, g_{\Delta,\ell}(u,v)\,, \tag{4.67}$$

where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$. The function g is a known function of Δ and ℓ. The dynamical information of the theory being studied is encoded in the Δs and the three-point couplings λ. The collection of $\{\Delta, \ell\}$ is called the CFT data. The conformal blocks in four dimensions can be written explicitly in terms of hypergeometric functions [93, 94],

$$g_{\Delta,\ell}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}(k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - z \leftrightarrow \bar{z})\,, \tag{4.68}$$

where $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$, and $k_{2\beta}(z) = z^{\beta}{}_2F_1(\beta, \beta, 2\beta, z)$. In the superconformal case a finite number of conformal families are related by supersymmetry transformations with known coefficients. This allows a rewriting of (4.67) in terms of a "superconformal block expansion",

$$\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle = \frac{1}{x_{12}^4 x_{34}^4} \sum_{\mathcal{O}\subset\mathcal{J}\times\mathcal{J}} \lambda^2_{\mathcal{J}\mathcal{J}\mathcal{O}}\, \mathcal{G}_{\Delta,\ell}(u,v)\,, \tag{4.69}$$

where the function $\mathcal{G}(u,v)$ is a superconformal block capturing the contributions of the superconformal multiplets appearing in (4.52), and it can be written as a finite sum of conformal blocks with coefficients fixed by supersymmetry

4.5.2 Toward the superconformal block

The contributions to the scalar four-point function $\langle JJJJ\rangle$ are quite limited, the operators have to be $SU(2)_R \times U(1)_r$ singlets and have even spin ℓ. We will now study the consequences of our selection rules (4.52). By scanning through the operator content of the different multiplets we can read which operators contribute to the expansion (4.69). Below we list our findings (the ranges for ℓ are given in section 3).

$$\begin{array}{lcll}
\mathcal{A}_{0,0(\frac{\ell}{2},\frac{\ell}{2})} & : & g_{\Delta,\ell} + b_1\, g_{\Delta+2,\ell+2} + b_2\, g_{\Delta+2,\ell} + b_3\, g_{\Delta+2,\ell-2} + b_4\, g_{\Delta+4,\ell} & \ell \quad \text{even} \\
\mathcal{A}_{0,0(\frac{\ell}{2},\frac{\ell}{2})} & : & g_{\Delta+1,\ell+1} + b_1\, g_{\Delta+1,\ell-1} + b_2\, g_{\Delta+3,\ell+1} + b_3\, g_{\Delta+3,\ell-1} & \ell \quad \text{odd} \\
\mathcal{A}_{0,0(\frac{\ell+2}{2},\frac{\ell}{2})} & : & g_{\Delta+2,\ell} + b_1\, g_{\Delta+2,\ell+2} & \ell \quad \text{even} \\
\mathcal{A}_{0,0(\frac{\ell+2}{2},\frac{\ell}{2})} & : & g_{\Delta+1,\ell+1} + b_1\, g_{\Delta+3,\ell+1} & \ell \quad \text{odd} \\
\mathcal{A}_{0,0(\frac{\ell+4}{2},\frac{\ell}{2})} & : & g_{\Delta+2,\ell+2} & \ell \quad \text{even} \\
\mathcal{C}_{0,-3(\frac{\ell}{2},\frac{\ell}{2})} & : & - & \\
\mathcal{C}_{\frac{1}{2},-\frac{3}{2}(\frac{\ell+1}{2},\frac{\ell}{2})} & : & g_{6+\ell,\ell} & \ell \quad \text{even} \\
\mathcal{C}_{\frac{1}{2},-\frac{3}{2}(\frac{\ell+1}{2},\frac{\ell}{2})} & : & g_{7+\ell,\ell+1} & \ell \quad \text{odd} \\
\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})} & : & g_{5+\ell,\ell-1} + b_1\, g_{5+\ell,\ell+1} + b_2\, g_{7+\ell,\ell+1} & \ell \quad \text{odd}
\end{array}$$

From this list we see that not all multiplets have a an associated superconformal block. Some of them contain several conformal families that contribute to the J correlator, others have one single family and the associated block is just bosonic, while one multiplet does not contribute at all. For the ones that do have superconformal blocks, the b_i coefficients need to be calculated.

One way to proceed would be a brute force calculation where the b_i couplings are extracted from our three-point functions. This procedure has been used for $\mathcal{N} = 1$ theories and its implementation for the $\mathcal{N} = 2$ case is just a straightforward generalization. However, due to the higher number of supercharges the calculation can become very cumbersome. Let us give an schematic outline of how the calculation goes for the $\mathcal{A}^{\Delta}_{0,0(\frac{\ell}{2},\frac{\ell}{2})}$ block, for more details we refer the reader to [118]. The starting point is the superspace expansion,

$$\begin{aligned}\mathcal{O}_{\alpha_1\cdots\alpha_\ell,\dot\alpha_1\cdots\dot\alpha_\ell} &= A_{\alpha_1\cdots\alpha_\ell,\dot\alpha_1\cdots\dot\alpha_\ell} + B^{j}_{i\,\alpha\alpha_1\cdots\alpha_\ell,\dot\alpha\dot\alpha_1\cdots\dot\alpha_\ell}\,\theta^{\alpha}_{j}\bar\theta^{\dot\alpha\,i} \\ &\quad + C^{jl}_{ik\,\alpha\beta\alpha_1\cdots\alpha_\ell,\dot\alpha\dot\beta\dot\alpha_1\cdots\dot\alpha_\ell}\,\theta^{\alpha}_{j}\bar\theta^{\dot\alpha\,i}\theta^{\beta}_{l}\bar\theta^{\dot\beta\,k} + \dots\end{aligned} \tag{4.70}$$

where $\alpha_1\cdots\alpha_\ell$ and $\dot\alpha_1\cdots\dot\alpha_\ell$ are symmetrized as usual. There are also terms proportional to $(\theta\bar\theta)^3$ and $(\theta\bar\theta)^4$ that contribute to this correlator but we will ignore them to avoid cluttering. Using the superconformal algebra (see appendix A) we can write:

$$B^{j}_{i\,\alpha\alpha_1\cdots\alpha_\ell,\dot\alpha\dot\alpha_1\cdots\dot\alpha_\ell} = \frac{1}{2}\Xi^{j}_{i\,\alpha\dot\alpha}A_{\alpha_1\cdots\alpha_\ell,\dot\alpha_1\cdots\dot\alpha_\ell}\,, \tag{4.71}$$

$$C^{jl}_{ik\,\alpha\beta\alpha_1\cdots\alpha_\ell,\dot\alpha\dot\beta\dot\alpha_1\cdots\dot\alpha_\ell} = \frac{1}{16}\Xi^{j}_{i\,\alpha\dot\alpha}\Xi^{l}_{k\,\beta\dot\beta}A_{\alpha_1\cdots\alpha_\ell,\dot\alpha_1\cdots\dot\alpha_\ell} - \frac{1}{4}\delta^{j}_{i}\delta^{l}_{k}\mathcal{P}_{\alpha\dot\alpha}\mathcal{P}_{\beta\dot\beta}A_{\alpha_1\cdots\alpha_\ell,\dot\alpha_1\cdots\dot\alpha_\ell}\,, \tag{4.72}$$

$$\vdots \tag{4.73}$$

where $\Xi^{j}_{i\,\alpha\dot\alpha} = [\mathcal{Q}^{j}_{\alpha}, \bar{\mathcal{Q}}_{i\,\dot\alpha}]$.

The next step is to build the conformal primaries associated to $B, C, \dots$ in order to obtain the three-point couplings that relate the different conformal families inside a multiplet. Once this is accomplished we can write,

$$\langle JJ\mathcal{O}\rangle \sim \langle JJA\rangle + \lambda_{JJB_{\rm prim}}\langle JJB_{\rm prim}\rangle\theta\bar\theta + \lambda_{JJC_{\rm prim}}\langle JJC_{\rm prim}\rangle(\theta\bar\theta)^2 + \dots \tag{4.74}$$

and the coefficients b_i can be calculated from,

$$\frac{\lambda^2_{JJB_{\rm prim}}}{N_{B_{\rm prim}}}\,, \qquad \frac{\lambda^2_{JJC_{\rm prim}}}{N_{C_{\rm prim}}}\,, \qquad \dots \tag{4.75}$$

where $N_X = \langle X|X\rangle$ is the norm of X. Although straightforward, the process becomes increasingly complicated the deeper one goes into the multiplet, i.e. the $(\theta\bar\theta)^3$ and $(\theta\bar\theta)^4$ terms.

$\mathcal{N}=1$ decomposition

Another way to organize the calculation is by splitting the $\mathcal{N}=2$ long multiplet in several $\mathcal{N}=1$ multiplets.[5] The idea is to organize the calculation in several $\mathcal{N}=1$ contributions and make full use of the $\mathcal{N}=1$ results already present in the literature. Let us start by decomposing an $\mathcal{N}=2$ multiplet in terms of $\mathcal{N}=1$ multiplets. The most efficient way to do this kind of decomposition is using superconformal characters [121, 137]. The expansion works as follows,

$$\begin{aligned}\mathcal{A}^{\Delta}_{0,0(\frac{\ell}{2},\frac{\ell}{2})} \sim \mathcal{A}^{\Delta}_{r_1=0(\frac{\ell}{2},\frac{\ell}{2})} &+ \mathcal{A}^{\Delta+1}_{r_1=0(\frac{\ell-1}{2},\frac{\ell-1}{2})} + \mathcal{A}^{\Delta+1}_{r_1=0(\frac{\ell+1}{2},\frac{\ell+1}{2})}\\ &+ \mathcal{A}^{\Delta+1}_{r_1=0(\frac{\ell-1}{2},\frac{\ell+1}{2})} + \mathcal{A}^{\Delta+1}_{r_1=0(\frac{\ell+1}{2},\frac{\ell-1}{2})} + \mathcal{A}^{\Delta+2}_{r_1=0(\frac{\ell}{2},\frac{\ell}{2})}\,,\end{aligned} \tag{4.76}$$

where $r_1 = \frac{2}{3}(r + 2\,\mathcal{R}^1{}_1)$ is the $\mathcal{N}=1$ r-charge after the decomposition. We have only written the $\mathcal{N}=1$ multiplets that have zero r_1-charge. Non-zero r_1-charge multiplets can not contribute to this correlator. From this expansion we conclude that only six $\mathcal{N}=1$ multiplets contribute to the OPE. Moreover, the highest dimension primary has $\Delta+2$, which means that the remaining $(\theta\bar\theta)^3$ and $(\theta\bar\theta)^4$ terms in (4.70) are $\mathcal{N}=1$ descendants, and

[5]We are indebted to Andy Stergiou for this idea.

therefore their contributions will be encoded in the $\mathcal{N} = 1$ results.

The superconformal blocks for $\mathcal{N} = 1$ conserved currents were worked out in [120, 119, 102]. Their results read,

$$G^{+}_{\Delta,\ell} = g_{\Delta,\ell} + \frac{(\Delta-2)^2(\Delta+\ell)(\Delta-\ell-2)}{16\Delta^2(\Delta+\ell+1)(\Delta-\ell-1)} g_{\Delta+2,\ell}\,, \tag{4.77}$$

$$G^{-}_{\Delta,\ell} = g_{\Delta+1,\ell-1} + \frac{(\ell+2)^2(\Delta+\ell+1)(\Delta-\ell-2)}{\ell^2(\Delta-\ell-1)(\Delta+\ell)} g_{\Delta+1,\ell-1}\,, \tag{4.78}$$

where $+(-)$ stands for ℓ even(odd). These results and the character identities imply that the $\mathcal{N} = 2$ superconformal block can be written as,

$$\mathcal{G}^{+}_{\Delta,\ell} = G^{+}_{\Delta,\ell} + c_1\, G^{-}_{\Delta+1,\ell-1} + c_2\, G^{-}_{\Delta+1,\ell+1} + c_3\, G^{+}_{\Delta+2,\ell} + c_4 g_{\Delta+2,\ell}\,, \tag{4.79}$$

$$\mathcal{G}^{-}_{\Delta,\ell} = G^{-}_{\Delta,\ell} + c_1\, G^{+}_{\Delta+1,\ell-1} + c_2\, G^{+}_{\Delta+1,\ell+1} + c_3\, G^{-}_{\Delta+2,\ell}\,. \tag{4.80}$$

The extra term in the even block represents the contributions of the $\mathcal{A}^{\Delta+1}_{r_1=0(\frac{\ell-1}{2},\frac{\ell+1}{2})}$ and $\mathcal{A}^{\Delta+1}_{r_1=0(\frac{\ell+1}{2},\frac{\ell-1}{2})}$ multiplets, which can not contribute to the ℓ odd block due to the scalar $J \times J$ selection rules. This decomposition simplifies significantly the analysis: the number of coefficients remains the same, but now we need to find primaries with dimensions up to $\Delta + 2$ instead of $\Delta + 4$.

The procedure sketched above is the same, but now we have to organize the calculation in superconformal primaries annihilated by the supercharges $\mathcal{S}_1^{\beta}$ and $\bar{\mathcal{S}}^{1\,\dot\beta}$ instead of conformal primaries annihilated by $\mathcal{K}^{\dot\beta\beta}$. Once this is accomplished we can write

$$\langle JJ\mathcal{O}\rangle \sim \langle JJA\rangle + \lambda_{JJB_{\rm sprim}}\langle JJB_{\rm sprim}\rangle\theta\bar\theta + \lambda_{JJC_{\rm sprim}}\langle JJC_{\rm sprim}\rangle(\theta\bar\theta)^2\,, \tag{4.81}$$

and the coefficients c_i can be calculated from,

$$\frac{\lambda^2_{JJB_{\rm sprim}}}{N_{B_{\rm sprim}}}\,, \qquad \frac{\lambda^2_{JJC_{\rm sprim}}}{N_{C_{\rm sprim}}}\,. \tag{4.82}$$

This is certainly a vast simplification of the problem, but still quite involved. Also, the $\mathcal{N}=1$ decomposition (4.76) works nicely for the $\mathcal{A}_{R=0,r=0(\frac{\ell}{2},\frac{\ell}{2})}$ multiplet but is not so efficient for the others. For example, it is significantly more complicated for the $\hat{\mathcal{C}}_{1(\frac{\ell}{2},\frac{\ell}{2})}$ multiplet.

4.A $\mathcal{N}=2$ superconformal algebra

In this appendix we collect our conventions for the $SU(2,2|2)$ algebra, we only list the non-vanishing commutators.

The conformal generators are $\{\mathcal{P}_{\alpha\dot\alpha},\, \mathcal{K}^{\alpha\dot\alpha},\, \mathcal{M}_\alpha{}^\beta,\, \bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta},\, D\}$. Lorentz indices transform canonically according to

$$[\mathcal{M}_\alpha{}^\beta, X_\gamma] = -2\delta^\beta_\alpha X_\alpha + \delta^\beta_\alpha X_\gamma\,, \qquad [\mathcal{M}_\alpha{}^\beta, X^\gamma] = 2\delta^\gamma_\alpha X^\beta - \delta^\beta_\alpha X^\gamma\,, \tag{4.83}$$

$$[\bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta}, X_{\dot\gamma}] = 2\delta^{\dot\alpha}_{\dot\gamma} X_{\dot\beta} - \delta^{\dot\alpha}_{\dot\beta} X_{\dot\gamma}\,, \qquad [\bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta}, X^{\dot\gamma}] = -2\delta^{\dot\gamma}_{\dot\beta} X^{\dot\alpha} + \delta^{\dot\alpha}_{\dot\beta} X^{\dot\gamma}\,. \tag{4.84}$$

The remaining $SO(4,2)$ commutators are,

$$[D,\mathcal{P}_{\alpha\dot\alpha}] = \mathcal{P}_{\alpha\dot\alpha}, \quad [D,\mathcal{K}^{\dot\alpha\alpha}] = -\mathcal{K}^{\dot\alpha\alpha} \tag{4.85}$$

and

$$[\mathcal{K}^{\dot\alpha\alpha},\mathcal{P}_{\beta\dot\beta}] = 2\delta^{\dot\alpha}_{\dot\beta}\mathcal{M}_\beta{}^\alpha - 2\delta^\alpha_\beta\bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta} - 4\delta^\alpha_\beta\delta^{\dot\alpha}_{\dot\beta} D\,. \tag{4.86}$$

The $SU(2)_R\times U(1)_r$ R-symmetry generators are denoted by $\{\mathcal{R}^i{}_j,\, r\}$. $SU(2)_R$ indices transform according to,

$$[\mathcal{R}^i{}_j, X_k] = -\delta^i_k X_j + \frac{1}{2}\delta^i_j X_k\,, \qquad [\mathcal{R}^i{}_j, X^k] = \delta^k_j X^i - \frac{1}{2}\delta^i_j X^k\,. \tag{4.87}$$

The fermionic generators are the Poincaré and conformal supercharges are $\{\mathcal{Q}^i_\alpha,\, \bar{\mathcal{Q}}_{i\,\dot\alpha}, \mathcal{S}^\alpha_i, \bar{\mathcal{S}}^{i\,\dot\alpha}\}$

and their anticommutators are,

$$\begin{aligned}
\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}_{j\,\dot\alpha}\} &= -2\mathrm{i}\delta^i_j\mathcal{P}_{\alpha\dot\alpha}, \quad \{\bar{\mathcal{S}}^{i\,\dot\alpha}, \mathcal{S}^\alpha_j\} = 2\mathrm{i}\delta^i_j\mathcal{K}^{\dot\alpha\alpha} &(4.88)\\
\{\mathcal{Q}^i_\alpha, \mathcal{S}^\beta_j\} &= -2\delta^i_j\mathcal{M}_\alpha{}^\beta + 2\delta^i_j\delta^\beta_\alpha\,(D-r) - 4\delta^\beta_\alpha\mathcal{R}^i{}_j &(4.89)\\
\{\bar{\mathcal{S}}^{i\,\dot\alpha},\bar{\mathcal{Q}}_{j\,\dot\beta}\} &= -2\delta^i_j\bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta} - 2\delta^i_j\delta^{\dot\alpha}_{\dot\beta}\,(D+r) - 4\delta^{\dot\alpha}_{\dot\beta}\mathcal{R}^i{}_j\,. &(4.90)
\end{aligned}$$

Finally, the commutators between bosonic and fermion generators,

$$[\mathcal{K}^{\dot\beta\beta}, \mathcal{Q}^i_\alpha] = -2\mathrm{i}\delta^\beta_\alpha\bar{\mathcal{S}}^{i\,\dot\beta}, \quad [\mathcal{K}^{\dot\beta\beta}, \bar{\mathcal{Q}}_{i\,\dot\alpha}] = 2\mathrm{i}\delta^{\dot\beta}_{\dot\alpha}S^\beta_i\,, \tag{4.91}$$

$$[\mathcal{P}_{\beta\dot\beta}, S^\alpha_i] = -2\mathrm{i}\delta^\alpha_\beta\bar{Q}_{i\,\dot\beta}, \quad [\mathcal{P}_{\beta\dot\beta}, \bar{\mathcal{S}}^{i\,\dot\alpha}]= 2\mathrm{i}\delta^{\dot\alpha}_{\dot\beta}\mathcal{Q}^i_\beta\,, \tag{4.92}$$

and

$$[D, \mathcal{Q}^i_\alpha] = \frac{1}{2}\mathcal{Q}^i_\alpha\,, \quad [D, \mathcal{S}^\alpha_i] = -\frac{1}{2}\mathcal{S}^\alpha_i\,, \quad [D, \bar{\mathcal{Q}}_{i\,\dot\alpha}] = \frac{1}{2}\bar{\mathcal{Q}}_{i\,\dot\alpha}\,, \qquad [D, \bar{\mathcal{S}}^{i\,\dot\alpha}] = -\frac{1}{2}\bar{\mathcal{S}}^{i\,\dot\alpha}\,, \tag{4.93}$$

$$[r, \mathcal{Q}^i_\alpha] = \frac{1}{2}\mathcal{Q}^i_\alpha\,, \quad [r, \mathcal{S}^\alpha_i] = -\frac{1}{2}\mathcal{S}^\alpha_i\,, \quad [r, \bar{\mathcal{Q}}_{i\,\dot\alpha}] = -\frac{1}{2}\bar{\mathcal{Q}}_{i\,\dot\alpha}\,, \quad [r, \bar{\mathcal{S}}^{i\,\dot\alpha}] = \frac{1}{2}\bar{\mathcal{S}}^{i\,\dot\alpha}\,. \tag{4.94}$$

4.B Superspace identities

Here we collect some superspace identities necessary for the three-point functions calculations of Section 3. Let us start proving that if equations (4.20) and the $\mathbb{Z}$ symmetry condition are satisfied, then equations (4.21) are also satisfied. In general the function H will always be expressible as sum of the form

$$H(\mathbf{Z}_3) = \sum_a f_a(\mathbf{X}_3)g_a(\Theta_3, \bar\Theta_3)\,, \tag{4.95}$$

where the functions g_a satisfy the conditions,

$$\frac{\partial^2}{\partial\Theta^i_{3\,\alpha}\partial\Theta^{\alpha\,j}_3}\,g_a(\Theta_3, \bar\Theta_3) = 0\,, \qquad \frac{\partial^2}{\partial\bar\Theta^{\dot\alpha}_{3\,i}\partial\bar\Theta_{3\,\dot\alpha\,j}}\,g_a(\Theta_3, \bar\Theta_3) = 0\,, \tag{4.96}$$

and $g_a(-\Theta_3, -\bar{\Theta}_3) = (-1)^F g_a(\Theta_3, \bar{\Theta}_3)$ where $(-1)^F$ counts the number of Θs and $\bar{\Theta}$s. From (4.22) it follows,

$$\mathcal{D}_{\alpha i}\,\bar{\mathbf{X}}_{3\,\alpha\dot{\alpha}} = 0\,, \qquad \tilde{\mathcal{D}}^i_{\dot{\alpha}}\,\bar{\mathbf{X}}_{3\,\alpha\dot{\alpha}} = 0\,, \tag{4.97}$$

and,

$$\begin{aligned}\mathcal{D}_{\alpha i}\mathcal{D}^\alpha_j\, H(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3) &= \mathcal{D}_{\alpha i}\mathcal{D}^\alpha_j \sum_a f_a(\mathbf{X}_3)\, g_a(\Theta_3, \bar{\Theta}_3)\\ &= \sum_a f_a(-\bar{\mathbf{X}}_3)\mathcal{D}_{\alpha i}\mathcal{D}^\alpha_j\, g_a(-\Theta_3, -\bar{\Theta}_3)\\ &= \sum_a f_a(-\bar{\mathbf{X}}_3)(-1)^F \frac{\partial^2}{\partial\Theta^i_{3\,\alpha}\partial\Theta^{\alpha\,j}_3}\, g_a(\Theta_3, \bar{\Theta}_3)\\ &= 0\end{aligned} \tag{4.98}$$

as promised. Thanks to this property imposing the conservation constraint is now an algebraic exercise, Taylor expanding both sides of the $\mathbb{Z}_2$ equation and equating coefficients solves the problem. For the expansion of the denominators we use the following identity,

$$\begin{aligned}\frac{1}{(\bar{\mathbf{X}}^2)^\Delta} = \frac{1}{(\mathbf{X}^2)^\Delta} - 4\mathrm{i}\,\Delta\frac{(\Theta^{\alpha i}\mathbf{X}_{\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}_i)}{(\mathbf{X}^2)^{\Delta+1}} - 8\,\Delta(\Delta-1)\frac{(\Theta^{\alpha i}\mathbf{X}_{\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}_i)^2}{(\mathbf{X}^2)^{\Delta+2}} - 8\,\Delta\frac{(\Theta^{\alpha\beta}\mathbf{X}_{\alpha\dot{\alpha}}\mathbf{X}_{\beta\dot{\beta}}\bar{\Theta}^{\dot{\alpha}\dot{\beta}})}{(\mathbf{X}^2)^{\Delta+2}}\\ + \frac{32}{3}\mathrm{i}\,\Delta(\Delta^2-1)\frac{(\Theta^{\alpha i}\mathbf{X}_{\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}_i)^3}{(\mathbf{X}^2)^{\Delta+3}} + \frac{32}{3}\,\Delta^2(\Delta^2-1)\frac{(\Theta^{\alpha i}\mathbf{X}_{\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}_i)^4}{(\mathbf{X}^2)^{\Delta+4}}\,,\end{aligned} \tag{4.99}$$

which for the special case $\Delta = 1$ becomes eq. (3.27) in [111]. After Taylor expanding what remains is to write all terms in our equation using the same basis of Grassmann objects. As usual with fermions, high powers of Grassmann variables are not all independent, for example,

$$(\Theta^{\alpha i}\mathbf{X}_{\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}_i)^3 = (\Theta^{\alpha\beta}\mathbf{X}_{\alpha\dot{\alpha}}\mathbf{X}_{\beta\dot{\beta}}\bar{\Theta}^{\dot{\alpha}\dot{\beta}})(\Theta^{\gamma\,i}\mathbf{X}_{\gamma\dot{\gamma}}\bar{\Theta}^{\dot{\gamma}}_i)\,. \tag{4.100}$$

Several identities of this type were needed for the calculations of section 3, we implemented them in Mathematica using the grassmannOps.m package by J. Michelson and M. Headrick.

Chapter 5

Mixed OPEs in $\mathcal{N} = 2$ superconformal theories

5.1 Introduction

Lagrangian methods seem to be insufficient when studying $\mathcal{N} = 2$ SCFTs. Although a large class of them are Lagrangian theories [41], there are many strongly coupled fixed points which seem to not allow a Lagrangian description [39, 40]. With the goal of developing a Lagrangian-free framework based only on the operator algebra, in [42] the conformal bootstrap program for $\mathcal{N} = 2$ theories was initiated. The conformal bootstrap [4, 5, 6] has received renewed attention after the work of [9]. The idea behind this approach is simple: imposing only unitarity and crossing symmetry for the four-point function, several CFT quantities can be obtained.

As pointed out in [42], there are three classes of short representations which are directly related to physical characteristics of $\mathcal{N} = 2$ theories, and thus can be regarded as a natural first step in the bootstrap program: the stress-tensor multiplet, the $\mathcal{N} = 2$ chiral multiplets and the flavor current multiplet. By bootstrapping them, we expect to obtain relevant information about the a and c anomalies, the Coulomb branch, the Higgs branch and the flavor central charge k, among other relations. Following this election of multiplets, they studied the four-point function of chiral operators and the four-point function of flavor current multiplets, obtaining several numerical bounds. There was a technical reason why the stress-tensor four point function was not studied in [42]: its conformal block expansion is not known. The block expansion for mixed operators is even more elusive.

Although the conformal block (or partial wave) decomposition of the four-point function is an essential ingredient in the conformal bootstrap program, there is no unified framework to compute the conformal blocks for different types of operators. Harmonic superspace techniques have proved useful to obtain the superconformal block expansion of $\frac{1}{2}$-BPS operators [117, 106], such as the flavor current multiplet. For the four-point function of two chiral and two anti-chiral operators, instead of harmonic superspace, chiral superspace has proven more useful [101]. The stress-tensor multiplet is not $\frac{1}{2}$-BPS nor chiral, but rather "semi-short" according to the classification of [109]. A first step towards its block decomposition was taken in [47], where, using standard Minkowski superspace techniques, the complete OPE of two stress-tensor multiplets was obtained. Due to the different nature of the three multiplets which we want to study in this article, $\mathcal{N} = 2$ Minkowski superspace seems suitable when dealing with a mixed combination of them. We denote the corresponding operators of the stress-tensor, the chiral and the flavor current multiplets as $\mathcal{J}$, Φ and $\mathcal{L}_{ij}$, respectively.

Another source of information used in [42] was the existence of a protected subsector of operators, present in every $\mathcal{N} = 2$ theory, that are isomorphic to a two-dimensional chiral algebra [92]. Using this correspondence between $\mathcal{N} = 2$ theories and chiral algebras, along with the block decomposition of the flavor current four-point function, bounds involving the central charge c and the flavor central charge k were obtained [92]. Following the same spirit, and using the $\mathcal{J} \times \mathcal{J}$ OPE, bounds to the central c were obtained [135]. Furthermore, studying mixed correlators in the chiral algebra setup, yet another bound relating c and k was obtained [138]. As pointed out in [138], an interesting result is obtained when combining the aforementioned analytical bounds involving both c and k: all the canonical rank one SCFTs associated to maximal mass deformations of the Kodaira singularities with flavor symmetry $G = A_1, A_2, D_4, E_6, E_7, E_8$ [38, 139, 140, 141] live at the intersection of the analytical bounds. It was also shown that the predicted theories with flavor symmetry $G = G_2, F_4$ [42], which have no known corresponding SCFT, live at the intersection of the bounds as well.

Those previous results entice us to keep studying systems of mixed correlators. While the single correlator bootstrap has already given interesting results, the addition of mixed correlators will give us access to the canonical rank one CFTs that live at the intersection of the analytical bounds. With the numerical bootstrap for the mixed system we will be able to explore CFT data inaccessible from the chiral algebra. Here we take a first step towards the construction of the superconformal blocks by obtaining the system of mixed OPE containing the three multiplets mentioned above: the stress-tensor multiplet, the chiral multiplets and the flavor current multiplet.

The outline of this article is as follows. In Section 2 we review $\mathcal{N} = 2$ superconformal three-point function, presenting all the ingredients needed to solve the OPE. In Section 3, after introducing the EOMs and conservation equations of the $\mathcal{J}$, Φ and $\mathcal{L}_{i\,j}$ superfields we show how to solve,

$$\langle \Phi\, \mathcal{J}\, \mathcal{O} \rangle\,, \qquad \langle \Phi\, \mathcal{L}_{i\,j}\, \mathcal{O} \rangle\,, \qquad \langle \mathcal{J}\, \mathcal{L}_{i\,j}\, \mathcal{O} \rangle\,, \tag{5.1}$$

for every $\mathcal{O}$ operator. This information allows us to write down the $\Phi \times \mathcal{J}$, $\Phi \times \mathcal{L}_{i\,j}$ and $\mathcal{J} \times \mathcal{L}_{i\,j}$ mixed OPEs. We end in Section 4 with conclusions. We also provide two appendices for notations and convention, plus solutions to the $\mathcal{O}$ operators not listed in the OPEs.

5.2 The three-point function of $\mathcal{N} = 2$ SCFT

It is well known that conformal symmetry fixes, up to an overall constant, the two- and three-point function for any operator. For a recent review see [8]. Superconformal symmetry also imposes restrictions to the form of the two- and three-point functions [112, 113]. The general expression for three-point functions in $\mathcal{N}=2$ superspace is,[1]

$$\langle \mathcal{O}^{(1)}_{\mathcal{I}_1}(z_1)\mathcal{O}^{(2)}_{\mathcal{I}_2}(z_2)\mathcal{O}^{(3)}_{\mathcal{I}_3}(z_3)\rangle = \frac{T^{(1)}_{\mathcal{I}_1}{}^{\mathcal{J}_1}\left(\hat{u}(z_{13})\right) T^{(2)}_{\mathcal{I}_2}{}^{\mathcal{J}_2}\left(\hat{u}(z_{23})\right)}{(x_{\bar{1}3})^{2\bar{q}_1}(x_{\bar{3}1})^{2q_1}(x_{\bar{2}3})^{2\bar{q}_2}(x_{\bar{3}2})^{2q_2}} H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}\left(\mathbf{Z}_3\right) \,, \tag{5.2}$$

where $z^A=(x^a,\theta^\alpha_i,\bar{\theta}^{i\,\dot{\alpha}})$ is the supercoordinate, q and $\bar{q}$ are given by $\Delta=q+\bar{q}$ and $r=q-\bar{q}$, r being the $U(1)_r$-charge. The $\mathcal{I}=(\alpha,\dot{\alpha},R,r)$ is a collective index that labels all the irreducible representation to which $\mathcal{O}$ belongs, it can also contain flavor indices. $H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}$ transforms as a tensor at z_3 in such a way that (5.2) is covariant. The chiral and anti-chiral coordinates are,

$$x^{\dot{\alpha}\alpha}_{\bar{1}2} = -x^{\dot{\alpha}\alpha}_{2\bar{1}} = x^{\dot{\alpha}\alpha}_{1-} - x^{\dot{\alpha}\alpha}_{2+} - 4\mathrm{i}\,\theta^\alpha_{2\,i}\bar{\theta}^{\dot{\alpha}i}_1 \,, \tag{5.3}$$

$$\theta_{12}=\theta_1-\theta_2\,, \qquad \bar{\theta}_{12}=\bar{\theta}_1-\bar{\theta}_2\,, \tag{5.4}$$

with $x^{\dot{\alpha}\alpha}_{\pm}=x^{\dot{\alpha}\alpha}\mp 2\mathrm{i}\theta^\alpha_i\bar{\theta}^{\dot{\alpha}\,i}$. The $\hat{u}$ matrices are defined as,

$$\hat{u}_i{}^j(z_{12}) = \left(\frac{x_{\bar{2}1}{}^2}{x_{\bar{1}2}{}^2}\right)^{1/2}\left(\delta^j_i - 4\mathrm{i}\frac{\theta_{12\,i}x_{\bar{1}2}\bar{\theta}^j_{12}}{x_{\bar{1}2}{}^2}\right) \,. \tag{5.5}$$

The argument of H is given by three superconformally covariant coordinates $\mathbf{Z}_3=(\mathbf{X}_3,\Theta_3,\bar{\Theta}_3)$, which are defined as,

$$\mathbf{X}_{3\,\alpha\,\dot{\alpha}} = \frac{x_{3\bar{1}\,\alpha\dot{\beta}}x^{\dot{\beta}\beta}_{\bar{1}2}x_{2\bar{3}\,\beta\dot{\alpha}}}{(x_{3\bar{1}})^2(x_{2\bar{3}})^2}\,, \qquad \bar{\mathbf{X}}_{3\,\alpha\dot{\alpha}} = \mathbf{X}^\dagger_{3\,\alpha\dot{\alpha}} = -\frac{x_{3\bar{2}\,\alpha\dot{\beta}}x^{\dot{\beta}\beta}_{\bar{2}1}x_{1\bar{3}\,\beta\dot{\alpha}}}{(x_{3\bar{2}})^2(x_{1\bar{3}})^2}\,, \tag{5.6}$$

$$\Theta^i_{3\,\alpha} = \mathrm{i}\left(\frac{x_{\bar{2}3\,\alpha\dot{\alpha}}}{x^2_{\bar{2}3}}\bar{\theta}^{\dot{\alpha}i}_{32} - \frac{x_{\bar{1}3\,\alpha\dot{\alpha}}}{x^2_{\bar{1}3}}\bar{\theta}^{\dot{\alpha}i}_{31}\right)\,, \qquad \bar{\Theta}_{3\,\dot{\alpha}\,i} = \mathrm{i}\left(\theta^\alpha_{32\,i}\frac{x_{\bar{3}2\,\alpha\dot{\alpha}}}{x^2_{\bar{3}2}} - \theta^\alpha_{31\,i}\frac{x_{\bar{3}1\,\alpha\dot{\alpha}}}{x^2_{\bar{3}1}}\right)\,. \tag{5.7}$$

An important relation which will play a key role in our computations is,

$$\bar{\mathbf{X}}_{3\,\alpha\dot{\alpha}} = \mathbf{X}_{3\,\alpha\dot{\alpha}} - 4\mathrm{i}\,\Theta^i_{3\,\alpha}\bar{\Theta}_{3\,\dot{\alpha}\,i}\,. \tag{5.8}$$

[1]We will follow the notation and conventions of [111], and we will also borrow some results from there.

In addition, the function H satisfies the scaling condition,

$$H^{\mathcal{I}}(\lambda\bar{\lambda}\mathbf{X}_3, \lambda\Theta_3, \bar{\lambda}\bar{\Theta}_3) = \lambda^{2a}\bar{\lambda}^{2\bar{a}} H^{\mathcal{I}}(\mathbf{X}_3, \Theta_3, \bar{\Theta}_3)\,, \tag{5.9}$$

with $a - 2\bar{a} = 2 - q$ and $\bar{a} - 2a = 2 - \bar{q}$. This last piece of information will help us identify the operator $\mathcal{O}^{(3)}$ by comparing its quantum numbers with all the possible representations listed in Tab. 5.A.

The conformally covariant operators $\mathcal{D}_A = \left(\partial/\partial\mathbf{X}_3^a, \mathcal{D}_{\alpha i}, \bar{\mathcal{D}}^{\dot{\alpha} i}\right)$ and $\mathcal{Q}_A = \left(\partial/\partial\mathbf{X}_3^a, \mathcal{Q}_{\alpha i}, \bar{\mathcal{Q}}^{\dot{\alpha} i}\right)$, given by,

$$\begin{aligned} \mathcal{D}_{\dot{\alpha} i} &= \frac{\partial}{\partial\Theta_3^{\alpha i}} + 4\mathrm{i}\bar{\Theta}_{3i}^{\dot{\alpha}} \frac{\partial}{\partial\mathbf{X}_3^{\dot{\alpha}\alpha}}\,, \qquad & \bar{\mathcal{D}}^{\dot{\alpha} i} &= \frac{\partial}{\partial\bar{\Theta}_{3\,\dot{\alpha} i}}\,, \\ \bar{\mathcal{Q}}^{\dot{\alpha} i} &= \frac{\partial}{\partial\bar{\Theta}_{3\,\dot{\alpha} i}} - 4\mathrm{i}\Theta_{3\,\alpha}^{i} \frac{\partial}{\partial\mathbf{X}_{3\,\alpha\dot{\alpha}}}\,, \qquad & \mathcal{Q}_{\alpha i} &= \frac{\partial}{\partial\Theta_3^{\alpha i}}\,, \end{aligned} \tag{5.10}$$

appear naturally when applying the superderivatives on $H\left(\mathbf{Z}\right)$:

$$\begin{aligned} D_{1\,\alpha}^{\,i} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right) &= \mathrm{i}\frac{(x_{\bar{3}\,1})_{\alpha\,\dot{\beta}}}{\left(x_{\bar{1}\,3}{}^2 x_{\bar{3}\,1}{}^2\right)^{1/2}} \hat{u}_j{}^i(z_{31}) \bar{\mathcal{D}}^{\dot{\beta} j} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right)\,, \\ \bar{D}_{1\,\dot{\beta}\,j} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right) &= \mathrm{i}\frac{(x_{\bar{1}\,3})_{\alpha\,\dot{\beta}}}{\left(x_{\bar{1}\,3}{}^2 x_{\bar{3}\,1}{}^2\right)^{1/2}} \hat{u}_j{}^i(z_{13}) \mathcal{D}_i^{\alpha} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right)\,, \\ D_{2\,\alpha}^{\,i} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right) &= \mathrm{i}\frac{(x_{\bar{3}\,2})_{\alpha\,\dot{\beta}}}{\left(x_{\bar{2}\,3}{}^2 x_{\bar{3}\,2}{}^2\right)^{1/2}} \hat{u}_j{}^i(z_{32}) \bar{\mathcal{Q}}^{\dot{\beta} j} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right)\,, \\ \bar{D}_{2\,\dot{\beta}\,j} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right) &= \mathrm{i}\frac{(x_{\bar{2}\,3})_{\alpha\,\dot{\beta}}}{\left(x_{\bar{2}\,3}{}^2 x_{\bar{3}\,2}{}^2\right)^{1/2}} \hat{u}_j{}^i(z_{23}) \mathcal{Q}_i^{\alpha} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right)\,. \end{aligned} \tag{5.11}$$

There are similar relations for quadratic derivatives. A quick computation shows,

$$D_1{}^{\alpha i} D_{1\,\alpha}^{\,j} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right) = -\frac{\hat{u}_k{}^i(z_{1\,3})\hat{u}_l{}^j(z_{1\,3})}{x_{\bar{1}\,3}{}^2 x_{\bar{3}\,1}{}^2} \bar{\mathcal{D}}_{\dot{\alpha}}^k \bar{\mathcal{D}}^{\dot{\alpha}\,l} H\left(\mathbf{X}_3, \Theta_3\,\bar{\Theta}_3\right)\,, \tag{5.12}$$

and similar relations for $\bar{D}_{1\,i\,j}$, $D_2{}^{i\,j}$ and $\bar{D}_{2\,i\,j}$. These relations will be very important when we impose the EOM/conservation equations of the superfields on the three-point function, restricting the form of all possible $\mathcal{O}$ in (5.2).

For a general CFT, the conformal symmetry is strong enough to to fix the OPE coefficients of the descendants in terms of that of the primary operator. This is not the case in supersymmetric theories, where nilpotent structures which contribute to the superdescendants can appear in the three-point function, see for example equation (3.23) in [111], and also equations (3.18) and (3.25) in [47]. In the cases studied here, the EOM/conservation equations will impose restrictions strong enough to fix the form of the three-point function completely, but this is not necessarily true for general operators.

5.3 Mixed OPE

We mentioned in the introduction that we are interested in the mixed OPEs of three multiplets: the stress-tensor multiplet, the $\mathcal{N}=2$ chiral multiplets and the flavor current multiplet, because of their close relation with physical properties of $\mathcal{N}=2$ theories:

• The semi-short multiplet $\hat{\mathcal{C}}_{0(0,0)}$[2] contains a conserved current of spin 2 and the spin 1 R-symmetry currents. It is well known that such spin 2 conserved current is the stress-tensor, which is present in every local theory, therefore, the study of this multiplet will give us general information about $\mathcal{N}=2$ theories. Its higher spin generalization $\hat{\mathcal{C}}_{0(j_1,j_2)}$ will contain higher spin conserved currents, which are not expected to appear in interacting theories [127, 128].

• The vacuum expectation value of chiral multiplets, $\mathcal{E}_q$,[3] parametrizes the Coulomb branch of the moduli space of $\mathcal{N}=2$ theories. The complex dimension of this branch defines the rank of the $\mathcal{N}=2$ theory. For a systematic study of rank one theories using their Coulomb branch geometries see [90, 143].

• The $\hat{\mathcal{B}}_1$ multiplet plays an important role in theories with flavor symmetry. Global symmetries currents can only appear in the $\hat{\mathcal{B}}_1$ or the $\hat{\mathcal{C}}_{0\left(\frac{1}{2},\frac{1}{2}\right)}$ multiplets. We already argued why this last multiplet must be absent. Therefore, for the study of $\mathcal{N}=2$ theories with flavor symmetries the $\hat{\mathcal{B}}_1$ multiplet is fundamental. In analogy with the relation between

[2] We will mostly follow the conventions of [109], see also Tab. 5.A for a summary.

[3] We define $\mathcal{E}_q := \mathcal{E}_{q(0,0)}$. Although chiral operator with higher spin, $\mathcal{E}_{q(j,0)}$ are allowed by representation theory, see Tab. 5.A, it was shown that such multiplets are absent in every known $\mathcal{N}=2$ theory [142].

chiral multiplets and the Coulomb branch, information about the Higgs branch is encoded in the $\hat{\mathcal{B}}_R$ multiplets.

As already noted, all of these multiplets are described by an $\mathcal{N} = 2$ superfield with a well known EOM/conservation equation. Indeed, the $\mathcal{N} = 2$ superspace conserved current associated to the stress-tensor, which we denote as $\mathcal{J}$, satisfies the conservation equations [144],

$$D^{i\,j}\mathcal{J}(z) = 0\,, \tag{5.13a}$$

$$\bar{D}^{i\,j}\mathcal{J}(z) = 0\,. \tag{5.13b}$$

The chiral multiplets $\mathcal{E}_q$ are described by an $\mathcal{N} = 2$ chiral superfield, denoted here by Φ, satisfying a linear equation,

$$D^{\dot{\alpha}\,i}\Phi(z) = 0\,, \tag{5.14}$$

which is the same for every r-charge. Unitarity implies $q \geq 1$. Because $\mathcal{E}_1$ is free, we will only consider the $q > 1$ cases. Finally, just as with the stress-tensor multiplet, the $\mathcal{N} = 2$ flavor current superfield, which we call $\mathcal{L}_{(i\,j)}$, satisfies the conservation equations,

$$D^{\alpha}_{(i}\mathcal{L}_{j\,k)}(z) = 0\,, \tag{5.15a}$$

$$\bar{D}^{\dot{\alpha}}_{(i}\mathcal{L}_{j\,k)}(z) = 0\,. \tag{5.15b}$$

Below, we solve the three-point function in order to obtain the OPE. We will first solve the OPE $\mathcal{E}_q \times \hat{\mathcal{C}}_{0(0,0)}$. The reason is twofold: first, it has been shown that a chiral field imposes a very strong constraint to the form of the three-point function, see for example [118, 42, 135]; second, since $\mathcal{J}$ carries no indices, possible solutions to the three-point function are, a priori, simpler than solutions with $\mathcal{L}_{i\,j}$. The solutions of $H\left(\mathbf{Z}\right)^{\mathcal{I}}$ tell us the quantum numbers of $\mathcal{O}^{\mathcal{I}}$. Knowledge of the quantum numbers allows us to identify the $\mathcal{O}^{\mathcal{I}}$ multiplet with the corresponding long, short or semi-short multiplet, see Tab. 5.A. Following this logic we next solve the OPE $\mathcal{E}_q \times \hat{\mathcal{B}}_1$. We end this section with the $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{B}}_1$ OPE.

5.3.1 $\mathcal{E}_q \times \hat{\mathcal{C}}_{0(0,0)}$

The three-point function (5.2) for a chiral operator and the stress tensor multiplet is,

$$\langle \Phi(z_1)\, \mathcal{J}(z_2)\, \mathcal{O}^I(z_3)\rangle = \frac{\lambda_{\Phi\mathcal{J}\mathcal{O}}}{(x_{\bar{3}1}^2)^q x_{\bar{2}3}^2 x_{2\bar{3}}^2} H^I(\mathbf{Z}_3)\,. \tag{5.16}$$

The EOM of Φ (5.14) and the conservation equation of $\mathcal{J}$ (5.13) will imply restrictions in the form of conformally covariant operators acting on $H\,(\mathbf{Z})$,

$$\bar{D}_1^{\dot{\alpha}\, i}\langle \Phi(z_1)\, \mathcal{J}(z_2)\, \mathcal{O}^I(z_3)\rangle = 0 \qquad \Rightarrow \qquad \mathcal{D}_{\alpha\, j}\, H^I(\mathbf{Z}_3) = 0\,, \tag{5.17}$$

$$D_2^{ij}\langle \Phi(z_1)\, \mathcal{J}(z_2)\, \mathcal{O}^I(z_3)\rangle = 0 \qquad \Rightarrow \qquad \bar{\mathcal{Q}}^k_{\dot{\alpha}}\, \bar{\mathcal{Q}}^{\dot{\alpha}\, l}\, H^I(\mathbf{Z}_3) = 0\,, \tag{5.18}$$

$$\bar{D}_2^{ij}\langle \Phi(z_1)\, \mathcal{J}(z_2)\, \mathcal{O}^I(z_3)\rangle = 0 \qquad \Rightarrow \qquad \mathcal{Q}^k_{\alpha}\, \mathcal{Q}^{\alpha\, l}\, H^I(\mathbf{Z}_3) = 0\,, \tag{5.19}$$

see (5.11). The $\mathcal{D}$ and $\mathcal{Q}$ operators were defined in (5.10).

The first constraint, (5.17), implies $H(\mathbf{X}, \Theta, \bar{\Theta}) = H(\mathbf{X} + 2\,\mathrm{i}\, \Theta\, \sigma\, \bar{\Theta}\,, \bar{\Theta}) = H(\bar{\mathbf{X}}\,, \bar{\Theta})$ (we omit the subscript 3 from now on.) Since $\bar{\mathcal{Q}}\, \bar{\mathbf{X}} = 0$, (5.18) implies that $H\left(\bar{\mathbf{X}}, \bar{\Theta}\right)$ can have at most a quadratic term in $\bar{\Theta}$ in the form of $\bar{\Theta}^{\dot{\alpha}}_i\, \bar{\Theta}^{\dot{\beta}\, i} = \bar{\Theta}^{\dot{\alpha}\,\dot{\beta}}$ [111]. Thus our solutions are of the form $H(\bar{\mathbf{X}}, \bar{\Theta}) = f(\bar{\mathbf{X}}) + g(\bar{\mathbf{X}})_{\dot{\alpha}\, i}\, \bar{\Theta}^{\dot{\alpha}\, i} + h(\bar{\mathbf{X}})_{\dot{\alpha}\,\dot{\beta}}\, \bar{\Theta}^{\dot{\alpha}\,\dot{\beta}}$. At this point is good to note that it is not possible, using only $\bar{\mathbf{X}}$, to construct functions f, g and h with any $SU(2)_R$-nor $U(1)_r$-charges. Therefore, we can, and will, study the solutions of the f, g and h terms separately. The constraint (5.19) implies,

$$\frac{\partial^2}{\partial\Theta^{\alpha\, i}\partial\Theta^{j}_{\alpha}} H(\mathbf{Z}) = -4\left(\bar{\Box} f(\bar{\mathbf{X}}) + \bar{\Box} g(\bar{\mathbf{X}})_{\dot{\alpha}\, k}\, \bar{\Theta}^{\dot{\alpha}\, k} + \bar{\Box} h(\bar{\mathbf{X}})_{\dot{\alpha}\,\dot{\beta}}\, \bar{\Theta}^{\dot{\alpha}\,\dot{\beta}}\right) \bar{\Theta}^{\dot{\mu}}_i\, \bar{\Theta}_{\dot{\mu}\, j} = 0\,, \tag{5.20}$$

where we defined $\bar{\Box} = \frac{\partial^2}{\partial\bar{\mathbf{X}}^a\partial\bar{\mathbf{X}}_a}$. A quick computation shows that $\bar{\Theta}^{\dot{\mu}}_i \bar{\Theta}_{\dot{\mu}\, j} \bar{\Theta}^{\dot{\alpha}\,\dot{\beta}}$ is always vanishing. This will generate solutions to (5.16) with arbitrary conformal dimension. We will identify those solutions with long supermultiplets. Furthermore, $\bar{\Theta}^{\dot{\mu}}_{i}\,\bar{\Theta}_{\dot{\mu}\,j}\,\bar{\Theta}^{\dot{\alpha}\,k}$ does not impose any new condition, thus, the solutions to $\bar{\square}f\left(\bar{\mathbf{X}}\right)=0$ are also solutions to $\bar{\square}g\left(\bar{\mathbf{X}}\right)=0$. The physical solutions to (5.20) are,

$$
\begin{aligned}
&\text{Multiplet} && H\left(\mathbf{Z}\right) \\
&\mathcal{A}^{\Delta}_{0,3-q\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}: && \left(\bar{\mathbf{X}}^2\right)^{-\frac{3}{2}+\frac{\Delta-\ell-q}{2}}\,\bar{\mathbf{X}}_{\alpha_1\,\dot{\alpha}_1}\cdots\bar{\mathbf{X}}_{\alpha_\ell\,\dot{\alpha}_\ell}\bar{\Theta}_{\dot{\alpha}_{\ell+1}\,\dot{\alpha}_{\ell+2}}\,, && (5.21a)\\
&\mathcal{A}^{\Delta}_{0,3-q,\left(\frac{\ell}{2},\frac{\ell}{2}\right)}: && \left(\bar{\mathbf{X}}^2\right)^{-\frac{3}{2}+\frac{\Delta-\ell-q}{2}}\,\bar{\mathbf{X}}_{\alpha_1\,\dot{\alpha}_1}\cdots\bar{\mathbf{X}}_{\alpha_{\ell-1}\,,\dot{\alpha}_{\ell-1}}\bar{\mathbf{X}}_{\alpha_\ell\,\dot{\mu}}\epsilon^{\dot{\mu}\,\dot{\beta}}\bar{\Theta}_{\dot{\alpha}_\ell\,\dot{\beta}}\,, && (5.21b)\\
&\mathcal{A}^{\Delta}_{0,3-q,\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)}: && \left(\bar{\mathbf{X}}^2\right)^{-\frac{5}{2}+\frac{\Delta-\ell-q}{2}}\,\bar{\mathbf{X}}_{\alpha_1\,,\dot{\alpha}_1}\cdots\bar{\mathbf{X}}_{\alpha_\ell\,,\dot{\alpha}_\ell}\bar{\mathbf{X}}_{\alpha_{\ell+1}\,\dot{\mu}}\bar{\mathbf{X}}_{\alpha_{\ell+2}\,\dot{\nu}}\bar{\Theta}^{\dot{\mu}\,\dot{\nu}}\,, && (5.21c)\\
&\bar{\mathcal{C}}_{0,-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)}: && \bar{\mathbf{X}}_{\alpha_1\,\dot{\alpha}_1}\cdots\bar{\mathbf{X}}_{\alpha_\ell\,\dot{\alpha}_\ell}\,, && (5.21d)\\
&\bar{\mathcal{C}}_{\frac{1}{2},\frac{3}{2}-q\left(\frac{\ell}{2},\frac{\ell+1}{2}\right)}: && \bar{\mathbf{X}}_{\alpha_1\,\dot{\alpha}_1}\cdots\bar{\mathbf{X}}_{\alpha_\ell\,\dot{\alpha}_\ell}\bar{\Theta}^{i}_{\dot{\alpha}_{\ell+1}}\,, && (5.21e)\\
&\mathcal{C}_{\frac{1}{2},-\frac{1}{2}\left(0,\frac{1}{2}\right)}: && \left(\bar{\mathbf{X}}^2\right)^{-1}\bar{\Theta}^{i}_{\dot{\alpha}}\,, && (5.21f)\\
&\bar{\mathcal{B}}_{\frac{1}{2},\frac{3}{2}-q\left(\frac{1}{2},0\right)}: && \left(\bar{\mathbf{X}}^2\right)^{-2}\bar{\mathbf{X}}_{\dot{\alpha}\,\dot{\mu}}\bar{\Theta}^{\dot{\mu}\,i}\,, && (5.21g)\\
&\bar{\mathcal{E}}_{-q(0,0)}: && \left(\bar{\mathbf{X}}\right)^{-1}\,. && (5.21h)
\end{aligned}
$$

There are also extra solutions to (5.16) which we have discarded, see (5.50) and (5.51).

When a long multiplet hits its unitarity bound, it decomposes into different (semi-)short multiplets [109]. The unitarity bounds of our three long multiplets (5.21a), (5.21b) and (5.21c) depend on the value of q. There are three distinctive ranges in every case. For (5.21a) its decomposition is,

$$
\begin{aligned}
q<2: &\quad \mathcal{A}^{5-q+\ell}_{0,3-q\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}\sim\mathcal{C}_{0,3-q\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\mathcal{C}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell-1}{2},\frac{\ell+2}{2}\right)}\,,\\
q=2: &\quad \mathcal{A}^{3+\ell}_{0,1\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}\sim\hat{\mathcal{C}}_{0\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell-1}{2},\frac{\ell+2}{2}\right)}+\hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell}{2},\frac{\ell+1}{2}\right)}+\hat{\mathcal{C}}_{1\left(\frac{\ell-1}{2},\frac{\ell+1}{2}\right)}\,, \qquad (5.22)\\
q>2: &\quad \mathcal{A}^{1+q+\ell}_{0,3-q\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}\sim\bar{\mathcal{C}}_{0,3-q\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\bar{\mathcal{C}}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell}{2},\frac{\ell+1}{2}\right)}\,.
\end{aligned}
$$

For (5.21b) the decomposition is,

$$
\begin{aligned}
q<3: &\quad \mathcal{A}^{5-q+\ell}_{0,3-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)} \sim \mathcal{C}_{0,3-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)} + \mathcal{C}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell-1}{2},\frac{\ell}{2}\right)}\,, \\
q=3: &\quad \mathcal{A}^{2+\ell}_{0,0\left(\frac{\ell}{2},\frac{\ell}{2}\right)} \sim \hat{\mathcal{C}}_{0\left(\frac{\ell}{2},\frac{\ell}{2}\right)} + \hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell-1}{2},\frac{\ell}{2}\right)} + \hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell}{2},\frac{\ell-1}{2}\right)} + \hat{\mathcal{C}}_{1\left(\frac{\ell-1}{2},\frac{\ell-1}{2}\right)}\,, \\
q>3: &\quad \mathcal{A}^{-1+q+\ell}_{0,3-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)} \sim \bar{\mathcal{C}}_{0,3-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)} + \bar{\mathcal{C}}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell}{2},\frac{\ell-1}{2}\right)}\,.
\end{aligned} \tag{5.23}
$$

Finally, (5.21c) decomposes as,

$$
\begin{aligned}
q<4: &\quad \mathcal{A}^{7-q+\ell}_{0,3-q\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} \sim \mathcal{C}_{0,3-q\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} + \mathcal{C}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell+1}{2},\frac{\ell}{2}\right)}\,, \\
q=4: &\quad \mathcal{A}^{3+\ell}_{0,-1\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} \sim \hat{\mathcal{C}}_{0\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} + \hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell+1}{2},\frac{\ell}{2}\right)} + \hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell+2}{2},\frac{\ell-1}{2}\right)} + \hat{\mathcal{C}}_{1\left(\frac{\ell+1}{2},\frac{\ell-1}{2}\right)}\,, \\
q>4: &\quad \mathcal{A}^{-1+q+\ell}_{0,3-q\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} \sim \bar{\mathcal{C}}_{0,3-q\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} + \bar{\mathcal{C}}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell+2}{2},\frac{\ell-1}{2}\right)}\,.
\end{aligned} \tag{5.24}
$$

Since our selection rules do not give any of the terms in the decomposition of the longs, we will follow the same procedure as in [47] and only take the first term of each decomposition in the OPE. The reason is simple: let us take, for example, the (5.21a) when $q < 2$. As we can see in (5.22), it decomposes into two semi-short multiplet: $\mathcal{C}_{0,3-q\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}$ and $\mathcal{C}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell-1}{2},\frac{\ell+2}{2}\right)}$. When we solve (5.16) imposing all the constraints (5.17), (5.18) and (5.19) we do not obtain any solution with the quantum numbers of $\mathcal{C}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell-1}{2},\frac{\ell+2}{2}\right)}$, therefore, our selection rules do not allow such multiplet as a solution to (5.16). The other multiplet in the expansion, $\mathcal{C}_{0,3-q\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}$, is nothing but a special limit of (5.21a), which is allowed by our selection rules. The reader might wonder if the selection rules ever allow a term in the decomposition of the long multiplet besides the first term, or maybe we are omitting valid solutions. Later, when studying the $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{B}}_1$ case, we will obtain a solution, (5.38j) and (5.38k), which appear in the decomposition of a long multiplet (5.38a).

Finally, we list the OPE between an $\mathcal{N}=2$ chiral and an $\mathcal{N}=2$ stress-tensor multiplet,

$$\begin{aligned}\mathcal{E}_{q(0,0)}\times\hat{\mathcal{C}}_{0(0,0)} \sim & \mathcal{E}_q+\mathcal{C}_{0,q\left(\frac{\ell}{2},\frac{\ell}{2}\right)}+\mathcal{C}_{\frac{1}{2},q-\frac{3}{2}\left(\frac{\ell+1}{2},\frac{\ell}{2}\right)}+\mathcal{B}_{\frac{1}{2},q-\frac{3}{2}\left(0,\frac{1}{2}\right)}+\mathcal{A}^{\Delta}_{0,q-3\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}\\ &+\mathcal{A}^{\Delta}_{0,q-3\left(\frac{\ell+1}{2},\frac{\ell+1}{2}\right)}+\mathcal{A}^{\Delta}_{0,q-3\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)}+\mathcal{F}_q\,,\end{aligned} \tag{5.25}$$

where $\mathcal{F}_q$ is,

$$\mathcal{F}_q=\begin{cases}\bar{\mathcal{C}}_{0,q-3\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\bar{\mathcal{C}}_{0,q-3\left(\frac{\ell+1}{2},\frac{\ell+1}{2}\right)}+\bar{\mathcal{C}}_{0,q-3\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} & q<2\\ \bar{\mathcal{C}}_{0,-1\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\bar{\mathcal{C}}_{0,-1\left(\frac{\ell+1}{2},\frac{\ell+1}{2}\right)}+\hat{\mathcal{C}}_{0\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)}+\bar{\mathcal{C}}_{\frac{1}{2},\frac{1}{2}\left(\frac{1}{2},0\right)} & q=2\\ \bar{\mathcal{C}}_{0,q-3\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\bar{\mathcal{C}}_{0,q-3\left(\frac{\ell+1}{2},\frac{\ell+1}{2}\right)}+\mathcal{C}_{0,q-3\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} & 3>q>2\\ \bar{\mathcal{C}}_{0,0\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\hat{\mathcal{C}}_{0\left(\frac{\ell+1}{2},\frac{\ell+1}{2}\right)}+\mathcal{C}_{0,0\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} & q=3\\ \bar{\mathcal{C}}_{0,q-3\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\mathcal{C}_{0,q-3\left(\frac{\ell+1}{2},\frac{\ell+1}{2}\right)}+\mathcal{C}_{0,q-3\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} & 4>q>3\\ \hat{\mathcal{C}}_{0\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\mathcal{C}_{0,1\left(\frac{\ell+1}{2},\frac{\ell+1}{2}\right)}+\mathcal{C}_{0,1\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} & q=4\\ \mathcal{C}_{0,q-3\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\mathcal{C}_{0,q-3\left(\frac{\ell+1}{2},\frac{\ell+1}{2}\right)}+\mathcal{C}_{0,q-3\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} & q>4\end{cases}, \tag{5.26}$$

and $\ell\geq 0$.

5.3.2 $\mathcal{E}_q\times\hat{\mathcal{B}}_1$

The three-point function (5.2) whith $\mathcal{O}^{(1)}=\Phi$ and $\mathcal{O}^{(2)}=\mathcal{L}_{ij}$ is given by,

$$\langle\Phi(z_1)\,\mathcal{L}_{ij}(z_2)\,\mathcal{O}^I(z_3)\rangle=\lambda_{\Phi\mathcal{L}\mathcal{O}}\frac{\hat{u}_i{}^k(z_{23})\hat{u}_j{}^l(z_{23})}{(x^2_{\bar{3}1})^q x^2_{\bar{2}3}x^2_{2\bar{3}}}H^I_{kl}(\mathbf{Z})\,, \tag{5.27}$$

where the $\hat{u}$ matrices were defined in (5.5). The symmetry $\mathcal{L}_{i\,j}=\mathcal{L}_{j\,i}$ must also appear in $H\,(\mathbf{Z})$, implying $H^{\mathcal{I}}_{mn}=H^{\mathcal{I}}_{nm}$.

Just as with $\mathcal{J}$, the conservation equations for $\mathcal{L}_{ij}$ (5.15) imply constraints to $H(\mathbf{Z})$,

$$D^{\alpha}_{(i}\,\mathcal{L}(z)_{j\,k)} = 0 \qquad \Rightarrow \qquad \bar{\mathcal{Q}}_{\dot\alpha\,(i}\,H_{mn)} = 0\,, \tag{5.28a}$$

$$\bar{D}^{\dot\alpha}_{(i}\,\mathcal{L}_{j\,k)}(z) = 0 \qquad \Rightarrow \qquad \mathcal{Q}_{\alpha\,(i}\,H_{mn)} = 0\,. \tag{5.28b}$$

Beside these conditions, we have the one that comes from the chiral supermultiplet (5.14), but we already know from (5.17) that it implies $H(\mathbf{Z})^{\mathcal{I}}_{mn} = H(\bar{\mathbf{X}},\bar\Theta)^{\mathcal{I}}_{mn}$.

Since $\bar{\mathcal{Q}}_{\dot\alpha\,i}\,\bar{\mathbf{X}}_{\mu\,\dot\mu} = 0$, we only need to expand $H(\bar{\mathbf{X}},\bar\Theta)$ in powers of $\bar\Theta$ and find which $\bar\Theta$ structures satisfy (5.28a). There are only three of those structures,

$$\epsilon^{(m|(i}\epsilon^{j)|n)}\,, \qquad \bar\Theta^{\dot\alpha(i}\epsilon^{j)m}\,, \qquad \bar\Theta^{(i\,j)}\,. \tag{5.29}$$

Finally, we use $\bar{\mathbf{X}}$ to construct all possible solutions to (5.28b). The solutions to (5.27) are,

$$\begin{aligned}
&\text{Multiplet} && H(\mathbf{Z}) \\
&\mathcal{A}^{\Delta}_{0,3-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)}: && \left(\bar{\mathbf{X}}^2\right)^{-\frac{3}{2}+\frac{\Delta-q-\ell}{2}}\,\bar{\mathbf{X}}_{\alpha_1\,\dot\alpha_1}\cdots\bar{\mathbf{X}}_{\alpha_\ell\,\dot\alpha_\ell}\bar\Theta^{i\,j}\,, && \text{(5.30a)}\\
&\bar{\mathcal{B}}_{1,-q(0,0)}: && \epsilon^{(m|(i}\epsilon^{j)|n)}\,, && \text{(5.30b)}\\
&\bar{\mathcal{C}}_{\frac{1}{2},\frac{3}{2}-q\left(\frac{\ell}{2},\frac{\ell+1}{2}\right)}: && \bar{\mathbf{X}}_{\alpha_1\,\dot\alpha_1}\cdots\bar{\mathbf{X}}_{\alpha_\ell\,\dot\alpha_\ell}\bar\Theta^{(i}_{\dot\alpha_{\ell+1}}\epsilon^{j)m}\,. && \text{(5.30c)}
\end{aligned}$$

For the only unphysical solution to (5.27) see (5.52)

The unitarity bound of our long multiplet (5.30a) depends on the $U(1)_r$-charge, in a similar fashion to the $\mathcal{E}_q \times \hat{\mathcal{C}}_{0(0,0)}$ case. Its decomposition is,

$$\begin{aligned}
q<3: &\quad \mathcal{A}^{5-q+\ell}_{0,3-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)} \sim \mathcal{C}_{0,3-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)} + \mathcal{C}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell-1}{2},\frac{\ell}{2}\right)}\,,\\
q=3: &\quad \mathcal{A}^{2+\ell}_{0,0\left(\frac{\ell}{2},\frac{\ell}{2}\right)} \sim \hat{\mathcal{C}}_{0\left(\frac{\ell}{2},\frac{\ell}{2}\right)} + \hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell-1}{2},\frac{\ell}{2}\right)} + \hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell}{2},\frac{\ell-1}{2}\right)} + \hat{\mathcal{C}}_{1\left(\frac{\ell-1}{2},\frac{\ell-1}{2}\right)}\,,\\
q>3: &\quad \mathcal{A}^{-1+q+\ell}_{0,3-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)} \sim \bar{\mathcal{C}}_{0,3-q\left(\frac{\ell}{2},\frac{\ell}{2}\right)} + \bar{\mathcal{C}}_{\frac{1}{2},\frac{5}{2}-q\left(\frac{\ell}{2},\frac{\ell-1}{2}\right)}\,.
\end{aligned} \tag{5.31}$$

Among the solutions that we found, there is no $\mathcal{C}_{\frac{1}{2},\frac{7}{2}-q\left(\frac{\ell-1}{2},\frac{\ell}{2}\right)}$, $\bar{\mathcal{C}}_{\frac{1}{2},\frac{5}{2}-q\left(\frac{\ell}{2},\frac{\ell-1}{2}\right)}$, $\hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell-1}{2},\frac{\ell}{2}\right)}$, $\hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell}{2},\frac{\ell-1}{2}\right)}$

nor $\hat{\mathcal{C}}_{1\left(\frac{\ell-1}{2},\frac{\ell-1}{2}\right)}$, therefore, we do not take them into account in the OPE, as explained before.

Finally, we list the OPE of $\mathcal{E}_{q(0,0)} \times \hat{\mathcal{B}}_1$,

$$\mathcal{E}_q \times \hat{\mathcal{B}}_1 \sim \mathcal{B}_{1,q(0,0)} + \mathcal{C}_{\frac{1}{2},q-\frac{3}{2}\left(\frac{\ell+1}{2},\frac{\ell}{2}\right)} + \mathcal{A}^{\Delta}_{0,q-3\left(\frac{\ell}{2},\frac{\ell}{2}\right)} + \begin{cases} \bar{\mathcal{C}}_{0,q-3\left(\frac{\ell}{2},\frac{\ell}{2}\right)} & q<3\,, \\ \hat{\mathcal{C}}_{0\left(\frac{\ell}{2},\frac{\ell}{2}\right)} & q=3\,, \\ \mathcal{C}_{0,q-3\left(\frac{\ell}{2},\frac{\ell}{2}\right)} & q>3\,. \end{cases} \tag{5.32}$$

With $\ell \geq 0$. Just as with the $\mathcal{E}_q \times \hat{\mathcal{C}}_{0(0,0)}$ OPE, the $\mathcal{E}_{q(0,0)} \times \hat{\mathcal{B}}_1$ OPE has a dependency on the value of q.

5.3.3 $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{B}}_1$

Our final mixed correlation function is between a stress-tensor multiplet and the flavor current. In this case, (5.2) reads,

$$\langle \mathcal{J}(z_1)\, \mathcal{L}_{ij}(z_2)\, \mathcal{O}^{\mathcal{I}}(z_3)\rangle = \lambda_{\mathcal{J}\mathcal{L}\mathcal{O}} \frac{\hat{u}_i{}^k(z_{23})\hat{u}_j{}^l(z_{23})}{x^2_{\bar{3}1}x^2_{\bar{2}3}x^2_{\bar{2}3}x^2_{2\bar{3}}} H^I_{kl}(\mathbf{Z})\,, \tag{5.33}$$

where the $\hat{u}$ matrices where defined in (5.5).

We already saw the implications of the conservation equations of $\mathcal{J}$ (5.13) and $\mathcal{L}$ (5.15) when we studied the $\mathcal{E}_q \times \hat{\mathcal{C}}_{0(0,0)}$ and $\mathcal{E}_q \times \hat{\mathcal{B}}_1$ OPE. The change of position of $\mathcal{J}$ from the second point to the first point only interchanges the $\mathcal{Q}$s for $\mathcal{D}$s,

$$D_1^{ij}\langle \mathcal{J}(z_1)\, \mathcal{L}_{ij}(z_2)\, \mathcal{O}^{\mathcal{I}}(z_3)\rangle = 0 \qquad \Rightarrow \qquad \bar{\mathcal{D}}^k_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha\, l} H^{\mathcal{I}}_{ij}(\mathbf{Z}_3) = 0\,, \tag{5.34a}$$

$$\bar{D}_1^{ij}\langle \mathcal{J}(z_1)\, \mathcal{L}_{ij}(z_2)\, \mathcal{O}^{\mathcal{I}}(z_3)\rangle = 0 \qquad \Rightarrow \qquad \mathcal{D}^k_{\alpha}\mathcal{D}^{\alpha\, l} H^{\mathcal{I}}_{ij}(\mathbf{Z}_3) = 0\,, \tag{5.34b}$$

The (5.34a) condition constraints $H_{kl}(\mathbf{Z})$ to be of the form [111, 47],

$$H^{\mathcal{I}}_{ij}(\mathbf{Z}) = f(\mathbf{X},\Theta)^{\mathcal{I}}_{ij} + g(\mathbf{X},\Theta)^{\mathcal{I}}_{ijk,\dot\alpha}\bar{\Theta}^{\dot\alpha\, k} + h(\mathbf{X},\Theta)^{\mathcal{I}}_{ij,\dot\alpha\dot\beta}\bar{\Theta}^{\dot\alpha\dot\beta}\,. \tag{5.35}$$

The next step is to find the f, g and h functions. Since (5.28b) does not mix the $\mathbf{X}$ with the Θ, it is natural to use this equation to construct the f, g and h terms as a Θ expansion,

$$f(\mathbf{X},\Theta)^{\mathcal{I}}_{(ij)} = \sum_{k=0}^{4} f_k(\mathbf{X})^{\mathcal{I}}_{(ij),m_1\cdots m_k,\alpha_1\cdots\alpha_k}\Theta^{\alpha_1 m_1}\cdots\Theta^{\alpha_k m_k}\,, \tag{5.36}$$

and similar for g and h. The following are the only terms satisfying (5.28b),

$$\begin{aligned}
&\epsilon^{(m|(i}\epsilon^{j)|n)}\,,\quad \epsilon^{m(i}\epsilon^{j)a}\,,\quad \epsilon^{(m|(i}\epsilon^{j)|n}\epsilon^{o)a}\,,\quad \Theta^{\alpha\,(m}\epsilon^{n)\,(i}\epsilon^{j)\,a} - \Theta^{\alpha\,a}\epsilon^{(m\,|(i}\epsilon^{j)|\,n)}\,,\\
&\Theta^{\alpha(i}\epsilon^{j)m}\,,\quad \Theta^{\alpha(i}\epsilon^{j)a}\,,\quad \Theta^{(i\,j)}\,,\qquad \Theta^{(i\,j)}\epsilon^{m\,a}\,.
\end{aligned} \tag{5.37}$$

Note that (5.37) contains the three structures in (5.29) plus five additional structures. The structures in (5.37) not only tell us the $SU(2)_R$-charge of the $\mathcal{O}$ operator in (5.33), but they also fix its $U(2)_r$-charge thanks to the scaling condition (5.9). The final step is to find the suitable functions of $\mathbf{X}$ in (5.36) satisfying both (5.28a) and (5.34b). The physical solutions are,

Multiplet	$H(\mathbf{Z})$

$$\mathcal{A}^{\Delta}_{0,0\left(\frac{\ell}{2},\frac{\ell}{2}\right)}:\quad \begin{aligned}
&-\frac{1}{2}(4+\ell-\Delta)\,\mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell)\dot\alpha_\ell)}\Theta^{\mu(i}\mathbf{X}_{\mu\dot\mu}\bar\Theta^{\dot\mu\,j)}\left(\mathbf{X}^2\right)^{-3+\frac{\Delta-\ell}{2}}\\
&+\mathrm{i}\,(2-\ell-\Delta)\,\mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot\alpha_{\ell-1}}\epsilon_{\dot\alpha_\ell)\dot\mu}\mathbf{X}_{\alpha_\ell)\dot\nu}\bar\Theta^{\dot\mu\,\dot\nu}\Theta^{ij}\left(\mathbf{X}^2\right)^{-3+\frac{\Delta-\ell}{2}}\\
&+(\Delta-2)\,\mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot\alpha_{\ell-1}}\Theta^{(i}_{\alpha_\ell)}\bar\Theta^{j)}_{\dot\alpha_\ell)}\left(\mathbf{X}^2\right)^{-2+\frac{\Delta-\ell}{2}}\,,
\end{aligned} \tag{5.38a}$$

$$\mathcal{A}^{\Delta}_{0,0\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}:\quad \begin{aligned}
&\mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell)\dot\alpha_\ell}\bigg(\frac{\mathrm{i}}{2}(2-\ell-\Delta)\,\bar\Theta_{\dot\alpha_{\ell+1}\dot\alpha_{\ell+1})}\Theta^{ij}\\
&+\Theta^{\mu(i}\mathbf{X}_{\mu(\dot\alpha_{\ell+1}}\bar\Theta^{j)}_{\dot\alpha_{\ell+2})}\bigg)\left(\mathbf{X}^2\right)^{-3+\frac{\Delta-\ell}{2}}\,,
\end{aligned} \tag{5.38b}$$

$$\mathcal{A}^{\Delta}_{0,0\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)}:\quad \begin{aligned}
&\mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell)\dot\alpha_\ell}\Big((6+\ell-\Delta)\,\mathbf{X}_{\alpha_{\ell+1}\dot\mu}\mathbf{X}_{\alpha_{\ell+2}\dot\nu}\bar\Theta^{\dot\mu\,\dot\nu}\Theta^{ij}\\
&-2\mathrm{i}\Theta^{(i}_{\alpha_{\ell+1}}\mathbf{X}_{\alpha_{\ell+2})\dot\mu}\bar\Theta^{\dot\mu\,j)}\Big)\left(\mathbf{X}^2\right)^{-4+\frac{\Delta-\ell}{2}}\,,
\end{aligned} \tag{5.38c}$$

$$\mathcal{C}_{0,0(0,1)}:\quad \Theta^{\mu(i}\mathbf{X}_{\mu(\dot\alpha_1}\bar\Theta^{j)}_{\dot\alpha_2)}\left(\bar{\mathbf{X}}^2\right)^{-2}\,, \tag{5.38d}$$

$$\bar{\mathcal{C}}_{\frac{1}{2},\frac{3}{2}\left(\frac{1}{2},0\right)}:\quad \mathbf{X}_{\alpha\,\dot\alpha}\bar\Theta^{\dot\alpha(i}\epsilon^{j)m}\left(\mathbf{X}^2\right)^{-2} - 4\mathrm{i}\mathbf{X}_{\alpha\,\dot\alpha}\mathbf{X}_{\beta\,\dot\beta}\bar\Theta^{\dot\alpha\,\dot\beta}\Theta^{\beta(i}\epsilon^{j)m}\left(\mathbf{X}^2\right)^{-3}\,, \tag{5.38e}$$

$$\mathcal{C}_{\frac{1}{2},-\frac{3}{2}\left(0,\frac{1}{2}\right)}: \qquad \Theta^{\mu(i}\mathbf{X}_{\mu\,\dot\alpha_1}\epsilon^{j)m}\left(\mathbf{X}^2\right)^{-2}\,, \tag{5.38f}$$

$$\mathcal{C}_{\frac{1}{2},\frac{3}{2}\left(0,\frac{1}{2}\right)}: \qquad \bar\Theta^{(i}_{\dot\alpha}\epsilon^{j)m}\,, \tag{5.38g}$$

$$\begin{aligned}\mathcal{C}_{\frac{1}{2},\frac{3}{2}\left(\frac{\ell}{2},\frac{\ell+1}{2}\right)}: \qquad & -2\mathrm{i}\ell\mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot\alpha_{\ell-1}}\bar\Theta_{\dot\alpha_\ell\dot\alpha_{\ell+1})}\Theta^{(i}_{\alpha_\ell)}\epsilon^{j)m}\\ & +\mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell}\bar\Theta^{(i}_{\dot\alpha_{\ell+1}}\epsilon^{j)m}\left(\mathbf{X}^2\right)\,,\end{aligned} \tag{5.38h}$$

$$\bar{\mathcal{C}}_{\frac{1}{2},-\frac{3}{2}\left(\frac{\ell+1}{2},\frac{\ell}{2}\right)}: \qquad \mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell)}\Theta^{(i}_{\alpha_{\ell+1})}\epsilon^{j)m}\,, \tag{5.38i}$$

$$\hat{\mathcal{C}}_{1(0,0)}: \qquad \epsilon^{(m|(i}\epsilon^{j)|n)}\,, \tag{5.38j}$$

$$\begin{aligned}\hat{\mathcal{C}}_{1\left(\frac{\ell}{2},\frac{\ell}{2}\right)}: \qquad & \mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot\alpha_{\ell-1}}\left(\mathbf{X}_{\alpha_\ell)\dot\alpha_\ell)}\epsilon^{(m|(i}\right.\\ & \left.-4\mathrm{i}\ell\left(\Theta^{(m}_{\alpha_\ell}\bar\Theta^{|(i}_{\dot\alpha_\ell)}+\Theta^{a}_{\alpha_\ell}\bar\Theta_{\dot\alpha_\ell)a}\epsilon^{(m|(i}\right)\right)\epsilon^{j)|n)}\,,\end{aligned} \tag{5.38k}$$

$$\hat{\mathcal{B}}_1: \qquad -4\mathrm{i}\mathbf{X}_{\mu\,\dot\mu}\left(\Theta^{\mu(n}\bar\Theta^{\dot\mu|(i}+\Theta^{\mu\,a}\bar\Theta^{\dot\mu}_{a}\epsilon^{(n|(i}\right)\epsilon^{j)|m)}\left(\mathbf{X}^2\right)^{-2}+\epsilon^{(n|(i}\epsilon^{j)|m)}\left(\mathbf{X}^2\right)^{-1}\,. \tag{5.38l}$$

The discarded solutions to (5.33) are listed in (5.53), (5.54) and (5.55)

The solution (5.38d) is exactly (5.38b) when it hits its unitarity bound, $\Delta_{UB} = 2$. It is also the only physical solution between a family of unphysical ones (5.53d). (5.38f) is also the only physical solution of a larger family (5.53a). (5.38h) is valid only for $\ell \geq 1$, the case $\ell = 0$ being (5.38g). A similar situation happens with (5.38k): it is only valid for $\ell \geq 1$, the special case $\ell = 0$ reduces to (5.38j), which is discarded.

Unlike the previous cases, the decomposition at the unitarity bound of the long multiplets in (5.38) are unique,

$$\mathcal{A}^{2+\ell}_{0,0\left(\frac{\ell}{2},\frac{\ell}{2}\right)} \sim \hat{\mathcal{C}}_{0\left(\frac{\ell}{2},\frac{\ell}{2}\right)}+\hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell-1}{2},\frac{\ell}{2}\right)}+\hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell}{2},\frac{\ell-1}{2}\right)}+\hat{\mathcal{C}}_{1\left(\frac{\ell-1}{2},\frac{\ell-1}{2}\right)}\,, \tag{5.39}$$

$$\mathcal{A}^{2+\ell}_{0,0\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)} \sim \mathcal{C}_{0,0\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)}+\mathcal{C}_{\frac{1}{2},\frac{1}{2}\left(\frac{\ell-1}{2},\frac{\ell+2}{2}\right)}\,, \tag{5.40}$$

$$\mathcal{A}^{2+\ell}_{0,0\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)} \sim \bar{\mathcal{C}}_{0,0\left(\frac{\ell+2}{2},\frac{\ell}{2}\right)}+\bar{\mathcal{C}}_{\frac{1}{2},-\frac{1}{2}\left(\frac{\ell+2}{2},\frac{\ell-1}{2}\right)}\,. \tag{5.41}$$

Since we do not find any $\hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell-1}{2},\frac{\ell+2}{2}\right)}$, $\hat{\mathcal{C}}_{\frac{1}{2}\left(\frac{\ell+2}{2},\frac{\ell-1}{2}\right)}$, $\mathcal{C}_{\frac{1}{2},\frac{1}{2}\left(\frac{\ell-1}{2},\frac{\ell+2}{2}\right)}$ nor $\bar{\mathcal{C}}_{-\frac{1}{2},-\frac{1}{2}\left(\frac{\ell+2}{2},\frac{\ell-1}{2}\right)}$ solutions, we discard them from the OPE. Note that the decomposition of (5.38a), (5.39), contains the

(5.38j,5.38k) solution, as discussed earlier.

Finally, we write down the $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{B}}_1$ OPE,

$$
\begin{aligned}
\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{B}}_1 \sim & \mathcal{C}_{0,0\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)} + \hat{\mathcal{C}}_{0\left(\frac{\ell+1}{2},\frac{\ell+1}{2}\right)} + \hat{\mathcal{C}}_{1\left(\frac{\ell}{2},\frac{\ell}{2}\right)} + \mathcal{C}_{\frac{1}{2},\frac{3}{2}\left(\frac{\ell}{2},\frac{\ell+1}{2}\right)} + \mathcal{C}_{\frac{1}{2},-\frac{3}{2}\left(0,\frac{1}{2}\right)} + \hat{\mathcal{B}}_1 \\
& + \mathcal{A}^{\Delta}_{0,0\left(\frac{\ell}{2},\frac{\ell+2}{2}\right)} + \mathcal{A}^{\Delta}_{0,0\left(\frac{\ell}{2},\frac{\ell}{2}\right)} ,
\end{aligned} \tag{5.42}
$$

with $\ell \geq 0$. Since this OPE is real, we do not write the conjugate multiplets.

5.4 Discussion

Using only $\mathcal{N}=2$ superconformal symmetry and Minkowski superspace techniques, we have computed the mixed OPE between a chiral multiplet $\mathcal{E}_q$, a stress-tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$, and a flavor current multiplet $\hat{\mathcal{B}}_1$. Those mixed OPEs were obtained by analyzing all possible three-point functions between two of the superfields corresponding to the multiplet listed before and an arbitrary third operator. The solutions were categorized as physical, which we listed in the OPEs (5.25), (5.32) and (5.42), and extra solutions listed in the Appendix B. The mixed OPEs involving an $\mathcal{E}_q$ multiplet have an explicit dependence on its $U(1)_r$-charge. This is not an unexpected result. Computation of two (anti-)chiral multiplets with different $U(1)_r$-charge also shows this behavior [135].

Our results are in complete agreement with the $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{C}}_{0(0,0)}$, $\mathcal{E}_q \times \bar{\mathcal{E}}_{-q}$ and $\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1$ OPEs previously found. Our mixed OPEs $\mathcal{E}_q \times \hat{\mathcal{C}}_{0(0,0)}$ and $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{B}}_1$ do not contain any $\hat{\mathcal{C}}_{0(0,0)}$ multiplet. This is in agreement with the $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{C}}_{0(0,0)}$ OPE [47], which does not contain any $\hat{\mathcal{B}}_1$ nor $\mathcal{E}_q$ operators. From the OPE between a chiral and an anti-chiral multiplet [117], it was expected to obtain a chiral multiplet $\mathcal{E}_q$ from the $\mathcal{E}_q \times \hat{\mathcal{C}}_{0(0,0)}$ OPE, and neither a $\hat{\mathcal{B}}_1$ nor $\mathcal{E}_q$ in the $\mathcal{E}_q \times \hat{\mathcal{B}}_1$ OPE. Finally, our $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{B}}_1$ OPE contains a $\hat{\mathcal{B}}_1$ multiplet in the expansion, which agrees with the $\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1$ OPE [101].

An interesting generalization of this work is to find the OPEs between different $\hat{\mathcal{B}}_R$ multiplet, with $R > 1$ and the $\hat{\mathcal{C}}_{0(0,0)}$ multiplet . As mentioned early, bounds for the central charge and the flavor central charge were obtained using the,

$$\langle \hat{\mathcal{C}}_{0(0,0)}\hat{\mathcal{C}}_{0(0,0)}\hat{\mathcal{C}}_{0(0,0)}\hat{\mathcal{C}}_{0(0,0)}\rangle\,, \qquad \langle \hat{\mathcal{B}}_1\hat{\mathcal{B}}_1\hat{\mathcal{B}}_1\hat{\mathcal{B}}_1\rangle \quad \text{and} \quad \langle \hat{\mathcal{C}}_{0(0,0)}\hat{\mathcal{C}}_{0(0,0)}\hat{\mathcal{B}}_1\hat{\mathcal{B}}_1\rangle\,, \tag{5.43}$$

correlators and the chiral algebra correspondence [92, 47, 138]. When those bounds are saturated, the OPE coefficients of certain allowed operators also vanish. For example, the bound that comes from the stress-tensor four-point function, $c \geq \frac{11}{30}$, is saturated only if the OPE coefficient of the $\hat{\mathcal{C}}_{1(0,0)}$ multiplet that appear in the $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{C}}_{0(0,0)}$ is 0. The theory with $c = \frac{11}{30}$ corresponds to the simplest Argyres-Douglas fixed point H_0. Using the superconformal index, it was confirmed that this multiplet does not appear in the $\hat{\mathcal{C}}_{0(0,0)} \times \hat{\mathcal{C}}_{0(0,0)}$ OPE in the H_0 theory [145]. By studying the OPEs involving a $\hat{\mathcal{B}}_R$ multiplet with higher $SU(2)_R$-charge, the chiral algebra correspondence should yield stronger bounds for the theory. Furthermore, the saturation of the bounds will imply the vanishing of certain OPE coefficient, as in the H_0 case. This vanishing of OPE coefficients can be given as input in the numerical bootstrap in order to single out a particular theory.

5.A Long, short and semi-short multiplets

Representation theory of the $\mathcal{N}=2$ superconformal algebra has been extensively used during this work. We follow the notation of [109], where all unitary irreducible representations of the extended superconformal algebra were constructed. The $\mathcal{N}=2$ superconformal algebra $SU(2,2|2)$ contains as a subalgebra the conformal algebra $SU(2,2)$ generated by $\{\mathcal{P}_{\alpha\dot{\alpha}}, \mathcal{K}_{\alpha\dot{\alpha}}, \mathcal{M}^{\alpha}_{\ \beta}, \bar{\mathcal{M}}^{\dot{\alpha}}_{\ \dot{\beta}}, \mathcal{D}\}$, where $\alpha = \pm$ and $\dot{\alpha} = \dot{\pm}$ are the Lorentz indices. $SU(2,2|2)$ also contains an R-symmetry algebra $SU(2)_R \times U(1)_r$ with generators $\{R^i_{\ j}, r\}$, where the $i, j = 1, 2$ are the $SU(2)_R$ indices. Along with the bosonic charges, there are fermionic supercharges, the Poincaré and conformal supercharges, $\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}\, i}, \mathcal{S}_i^{\alpha}, \bar{\mathcal{S}}^{\dot{\alpha}\, i}\}$.

The spectrum of operators of $SU(2,2|2)$ is constructed from its highest weight, or superconformal primary. Acting with the Poincaré supercharges on the superconformal primary, superconformal descendants are generated. A general superconformal primary is denoted by $\mathcal{A}^{\Delta}_{R,r(j,\bar{j})}$, and is referred to as long multiplet. The only restriction for such multiplet is to satisfy a unitary bound [146],

$$\Delta \geq 2 + 2j + 2R + r\,,\ 2 + 2\bar{j} + 2R - r\,. \tag{5.44}$$

If the multiplet is annihilated by a certain combination of $\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}\, i}\}$ is referred to as short or semi-short and it saturates the unitarity bound. These combinations are,

$$\mathcal{B}^i : \qquad \mathcal{Q}^i_\alpha \Psi = 0\,, \tag{5.45}$$

$$\bar{\mathcal{B}}^i : \qquad \bar{\mathcal{Q}}^i_{\dot{\alpha}} \Psi = 0\,, \tag{5.46}$$

$$\mathcal{C}^i : \qquad \begin{cases} \epsilon^{\alpha\beta} \mathcal{Q}^i_\alpha \Psi_\beta & j \neq 0\,, \\ \epsilon^{\alpha\beta} \mathcal{Q}^i_\alpha \mathcal{Q}^i_\beta \Psi & j = 0\,, \end{cases} \tag{5.47}$$

$$\bar{\mathcal{C}}^i : \qquad \begin{cases} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\mathcal{Q}}^i_{\dot{\alpha}} \Psi_{\dot{\beta}} & \bar{j} \neq 0\,, \\ \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\mathcal{Q}}^i_{\dot{\alpha}} \bar{\mathcal{Q}}^i_{\dot{\beta}} \Psi & \bar{j} = 0\,. \end{cases} \tag{5.48}$$

$\mathcal{B}$-type conditions are called short while $\mathcal{C}$-type are called semi-short, because the former are stronger conditions. In Tab. 5.A we list all possible shortening conditions.

The decomposition of a long multiplet $\mathcal{A}^{\Delta}_{R,r(j,\bar{j})}$ when it hits its unitarity bound is given by,

$$\begin{aligned} \mathcal{A}^{2+2j+2R+r}_{R,r(j,\bar{j})} &\sim \mathcal{C}_{R,r(j,\bar{j})} + \mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}\left(j-\frac{1}{2},\bar{j}\right)}\,, \\ \mathcal{A}^{2+2j+2\bar{j}+2R}_{R,r(j,\bar{j})} &\sim \hat{\mathcal{C}}_{R(j,\bar{j})} + \hat{\mathcal{C}}_{R+\frac{1}{2}\left(j-\frac{1}{2},\bar{j}\right)} + \hat{\mathcal{C}}_{R+\frac{1}{2}\left(j,\bar{j}-\frac{1}{2}\right)} + \hat{\mathcal{C}}_{R+1\left(j-\frac{1}{2},\bar{j}-\frac{1}{2}\right)}\,, \\ \mathcal{A}^{2+2\bar{j}+2R-r}_{R,r(j,\bar{j})} &\sim \bar{\mathcal{C}}_{R,r(j,\bar{j})} + \bar{\mathcal{C}}_{R+\frac{1}{2},r-\frac{1}{2}\left(j,\bar{j}-\frac{1}{2}\right)}\,. \end{aligned} \tag{5.49}$$

Shortening	Unitarity bounds		Multiplet
	$\Delta > 2 + 2j + 2R + r$	$\Delta > 2 + 2\bar{j} + 2R - r$	$\mathcal{A}^{\Delta}_{R,r(j,\bar{j})}$
$\mathcal{B}^1$	$\Delta = 2R + r$	$j = 0$	$\mathcal{B}_{R,r(0,\bar{j})}$
$\bar{\mathcal{B}}_2$	$\Delta = 2R - r$	$\bar{j} = 0$	$\bar{\mathcal{B}}_{R,r(j,0)}$
$\mathcal{B}^1 \cap \mathcal{B}^2$	$\Delta = r$	$R = \bar{j} = 0$	$\mathcal{E}_{r(0,\bar{j})}$
$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$\Delta = -r$	$R = j = 0$	$\bar{\mathcal{E}}_{r(j,0)}$
$\mathcal{B}^1 \cap \bar{\mathcal{B}}_2$	$\Delta = 2R$	$j = \bar{j} = r = 0$	$\hat{\mathcal{B}}_R$
$\mathcal{C}^1$	$\Delta = 2 + 2j + 2R + r$		$\mathcal{C}_{R,r(j,\bar{j})}$
$\bar{\mathcal{C}}_2$	$\Delta = 2 + 2\bar{j} + 2R - r$		$\bar{\mathcal{C}}_{R,r(j,\bar{j})}$
$\mathcal{C}^1 \cap \mathcal{C}^2$	$\Delta = 2 + 2j + r$	$R = 0$	$\mathcal{C}_{0,r(j,\bar{j})}$
$\bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$\Delta = 2 + 2\bar{j} - r$	$R = 0$	$\bar{\mathcal{C}}_{0,r(j,\bar{j})}$
$\mathcal{C}^1 \cap \bar{\mathcal{C}}_2$	$\Delta = 2 + j + \bar{j} + 2R$	$r = \bar{j} - j$	$\hat{\mathcal{C}}_{R(j,\bar{j})}$
$\mathcal{B}^1 \cap \bar{\mathcal{C}}_2$	$\Delta = 1 + \bar{j} + 2R$	$r = \bar{j} + 1$	$\mathcal{D}_{R(0,\bar{j})}$
$\bar{\mathcal{B}}_2 \cap \mathcal{C}^1$	$\Delta = 2 + j + 2R$	$-r = j + 1$	$\bar{\mathcal{D}}_{R(j,0)}$
$\mathcal{B}^1 \cap \mathcal{B}^2 \cap \bar{\mathcal{C}}_2$	$\Delta = r = 1 + \bar{j}$	$R = 0$	$\mathcal{D}_{0(0,\bar{j})}$
$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2 \cap \mathcal{C}^1$	$\Delta = -r = 1 + j$	$R = 0$	$\bar{\mathcal{D}}_{0(j,0)}$

Table 5.1: All possible short and semi-short representations for the $\mathcal{N} = 2$ superconformal algebra.

5.B Discarded solutions

Several solutions to (5.16), (5.27) and (5.33) were not listed in the corresponding OPE, because we regarded them as unphysical. We categorize them in three types,

- Non-unitary solutions. Those solutions have a conformal dimension below the unitarity bound corresponding to their quantum numbers.

- Long multiplets with fixed dimension. Long multiplets with fixed dimensions were argued to come from a theory with extended $\mathcal{N} = 4$ symmetry [47]. We are only interested in theories with $\mathcal{N} = 2$, thus, we will consider such multiplets as being irrelevant to $\mathcal{N} = 2$ dynamics.

- Solutions with a vanishing overall coefficient. We also find a case where the solution to the three-point function corresponds to a physical multiplet, a stress-tensor multiplet. Uniqueness of the stress-tensor implies another symmetry of the three-point function, which is only satisfied if the overall coefficient vanishes.

$\mathcal{E}_q \times \hat{\mathcal{C}}_{0(0,0)}$

There are two types of discarded solutions to (5.16) which are not listed in (5.25): non-unitary and solution corresponding to a long multiplet with fixed dimension. The non-unitary solutions are,

$$\begin{aligned}
(\Delta, R, r, j, \bar{j}) \qquad & H(\mathbf{Z}) \\
\left(\frac{1}{2}+q, \frac{1}{2}, \frac{3}{2}-q, 0, \frac{1}{2}\right): \qquad & \left(\bar{\mathbf{X}}^2\right)^{-1} \bar{\Theta}^i_{\dot{\alpha}}\,, && \text{(5.50a)} \\
\left(q-\ell, 0, -q, \frac{\ell}{2}, \frac{\ell}{2}\right): \qquad & \left(\bar{\mathbf{X}}^2\right)^{-1-\ell} \bar{\mathbf{X}}_{\alpha_1 \dot{\alpha}_1} \cdots \bar{\mathbf{X}}_{\alpha_\ell \dot{\alpha}_\ell} && \text{(5.50b)} \\
\left(-\frac{1}{2}+q-\ell, \frac{1}{2}, \frac{3}{2}-q, \frac{\ell+1}{2}, \frac{\ell}{2}\right): \qquad & \left(\bar{\mathbf{X}}^2\right)^{-2-\ell} \bar{\mathbf{X}}_{\alpha_1 \dot{\alpha}_1} \cdots \bar{\mathbf{X}}_{\alpha_\ell \dot{\alpha}_\ell} \bar{\mathbf{X}}_{\dot{\alpha}_{\ell+1}\, \dot{\mu}} \bar{\Theta}^{\dot{\mu}\, i}\,. && \text{(5.50c)}
\end{aligned}$$

Although it is puzzling to find solutions with a conformal dimension that decreases with the spin, this kind of solutions are not new. They have already appeared in $\mathcal{N}=1$ theories when computing the three-point function with two flavor currents [119] and in $\mathcal{N}=2$ theories when studying the three-point function with two stress-tensor multiplets [47].

The only long multiplet with fixed dimension is,

$$\mathcal{A}^{\frac{7}{2}+q+\ell}_{\frac{1}{2},\frac{3}{2}-q\left(\frac{\ell+1}{2},\frac{\ell}{2}\right)}: \qquad \bar{\mathbf{X}}_{\alpha_1\dot{\alpha}_1}\cdots\bar{\mathbf{X}}_{\alpha_\ell\dot{\alpha}_\ell}\bar{\mathbf{X}}_{\dot{\alpha}_{\ell+1}\dot{\mu}}\bar{\Theta}^{\dot{\mu}\,i}\,. \tag{5.51}$$

$\mathcal{E}_q\times\hat{\mathcal{B}}_1$

The only unphysical solution to (5.27) is,

$$\left(-\frac{3}{2}+q-\ell,\frac{1}{2},\frac{3}{2}-q,\frac{\ell+1}{2},\frac{\ell}{2}\right): \qquad \left(\bar{\mathbf{X}}^2\right)^{-2-\ell}\bar{\mathbf{X}}_{\alpha_1\dot{\alpha}_1}\cdots\bar{\mathbf{X}}_{\alpha_\ell\dot{\alpha}_\ell}\bar{\mathbf{X}}_{\alpha_{\ell+1}\dot{\beta}}\bar{\Theta}^{\dot{\beta}(i}\epsilon^{j)m}\,. \tag{5.52}$$

Since the conformal dimension of (5.52) is below the unitarity bound for its quantum numbers, $\Delta_{UB}=\frac{11}{2}-q+\ell$ for $4\geq q\geq 1$ and $\Delta_{UB}=\frac{3}{2}+q+\ell$ for $q\geq 4$, we regard it as a non-unitarity solution.

$\hat{\mathcal{C}}_{0(0,0)}\times\hat{\mathcal{B}}_1$

The non-unitary solutions to (5.33) are,

$$\begin{aligned}
&(\Delta,R,r,j,\bar{j}) && H(\mathbf{Z})\\
&\left(\frac{3}{2}-\ell,\frac{1}{2},-\frac{3}{2},\frac{\ell+1}{2},\frac{\ell}{2}\right): && \mathbf{X}_{(\alpha_1(\dot{\alpha}_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot{\alpha}_{\ell-1}}\Big(\mathbf{X}_{\alpha_\ell\dot{\alpha}_\ell)}\mathbf{X}_{\alpha_{\ell+1}\dot{\beta}}\bar{\Theta}^{\dot{\beta}(i}\epsilon^{j)m}\left(\mathbf{X}^2\right)^{-2-\ell}\\
& && -2\mathrm{i}\,(2+\ell)\,\mathbf{X}_{\alpha_\ell\dot{\alpha}_\ell)}\mathbf{X}_{\alpha_{\ell+1})\dot{\beta}}\mathbf{X}_{\mu\dot{\mu}}\bar{\Theta}^{\dot{\beta}\dot{\mu}}\Theta^{\mu(i}\epsilon^{j)m}\left(\mathbf{X}^2\right)^{-3-\ell}\\
& && -2\mathrm{i}\ell\epsilon_{\dot{\alpha}_\ell\dot{\beta}}\mathbf{X}_{\alpha_\ell)\,\dot{\mu}}\bar{\Theta}^{\dot{\mu}\,\dot{\beta}}\Theta^{(i}_{\alpha_{\ell+1})}\epsilon^{j)m}\left(\mathbf{X}^2\right)^{-2-\ell}\Big)\,, && (5.53\text{a})\\
&\left(1-\ell,0,0,\frac{\ell}{2},\frac{\ell}{2}\right): && \mathbf{X}_{(\alpha_1(\dot{\alpha}_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot{\alpha}_{\ell-1}}\Big(\ell\Theta^{(i}_{\alpha_\ell)}\bar{\Theta}^{j)}_{\dot{\alpha}_\ell)}\left(\mathbf{X}^2\right)^{-1-\ell}\\
& && (1+\ell)\mathbf{X}_{\alpha_\ell)\dot{\alpha}_\ell)}\Theta^{\mu(i}\mathbf{X}_{\mu\dot{\mu}}\bar{\Theta}^{\dot{\mu}\,j)}\left(\mathbf{X}^2\right)^{-2-\ell}\Big) && (5.53\text{b})
\end{aligned}$$

$$\left(2,0,0,\frac{\ell}{2},\frac{\ell}{2}\right): \qquad \mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot\alpha_{\ell-1}}\left(\mathbf{X}_{\alpha_\ell)\dot\alpha_\ell)}\Theta^{\mu(i}\mathbf{X}_{\mu\dot\mu}\bar\Theta^{\dot\mu j)}\right.$$

$$\left.-\mathrm{i}\ell\Theta^{ij}\epsilon_{\dot\alpha_\ell)\dot\mu}\mathbf{X}_{\alpha_\ell\dot\nu}\bar\Theta^{\dot\mu\dot\nu}\right)\left(\mathbf{X}^2\right)^{-(4+\ell)/2}, \tag{5.53c}$$

$$\left(2-\ell,0,0,\frac{\ell}{2},\frac{\ell+2}{2}\right): \qquad \mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell)\dot\alpha_\ell)}\Theta^{\mu(i}\mathbf{X}_{\mu\dot\alpha_{\ell+1}}\bar\Theta^{j)}_{\dot\alpha_\ell)}\left(\bar{\mathbf{X}}^2\right)^{-2-\ell}, \tag{5.53d}$$

$$\left(2-\ell,1,0,\frac{\ell}{2},\frac{\ell}{2}\right): \qquad \mathbf{X}_{(\alpha_1\dot\alpha_1}\cdots\mathbf{X}_{\alpha_{\ell-1}\dot\alpha_{\ell-1}}\left(\mathbf{X}_{\alpha_\ell)\dot\alpha_\ell)}\epsilon^{(m|(i}\epsilon^{j)|n)}\left(\mathbf{X}^2\right)\right.$$

$$-4\mathrm{i}(\ell+1)\mathbf{X}_{\alpha_\ell)\dot\alpha_\ell)}\mathbf{X}_{\mu\dot\mu}\left(\Theta^{\mu(m}\bar\Theta^{\dot\mu|(i}+\Theta^{\mu a}\bar\Theta^{\dot\mu}_{a}\epsilon^{(m|(i}\right)\epsilon^{j)|n)}$$

$$\left.-4\ell\left(\Theta^{(m}_{\alpha_\ell)}\bar\Theta^{|(i}_{\dot\alpha_\ell)}+\Theta^{a}_{\alpha_\ell)}\bar\Theta_{\dot\alpha_\ell)a}\epsilon^{(m|(i}\right)\epsilon^{j)|n)}\left(\mathbf{X}^2\right)\right)\left(\mathbf{X}^2\right)^{-2-\ell}, \tag{5.53e}$$

$$\left(\frac{3}{2}-\ell,\frac{1}{2},-\frac{3}{2},\frac{\ell}{2},\frac{\ell+1}{2}\right): \qquad \mathbf{X}_{(\alpha_1\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell)\dot\alpha_\ell}\Theta^{\mu(i}\mathbf{X}_{\mu\dot\alpha_{\ell+1}}\epsilon^{j)m}\left(\mathbf{X}^2\right)^{-\ell-2}. \tag{5.53f}$$

We also find a solution to (5.33) which corresponds to a long multiplet with fixed dimension,

$$\mathcal{A}^{6+\ell}_{0,0\left(\frac{\ell}{2},\frac{\ell}{2}\right)}: \qquad \mathbf{X}_{(\alpha_1(\dot\alpha_1}\cdots\mathbf{X}_{\alpha_\ell\dot\alpha_\ell)}\Theta^{(i}_{\alpha_{\ell+1}}\mathbf{X}_{\alpha_{\ell+2})\dot\mu}\bar\Theta^{\dot\mu j)}. \tag{5.54}$$

As explained before, we regard this solution as coming from a theory with enhanced $\mathcal{N}=4$ symmetry.

Finally, there is a very special solution to (5.33),

$$H(\mathbf{Z})=\Theta^{\alpha(i}\mathbf{X}_{\alpha\dot\alpha}\bar\Theta^{\dot\alpha j)}\left(\mathbf{X}^2\right)^{-2}, \tag{5.55}$$

which has conformal dimension $\Delta=2$. This solution belongs to a $\hat{\mathcal{C}}_{0(0,0)}$ multiplet, corresponding to a stress-tensor multiplet. Studying the $\hat{\mathcal{C}}\times\hat{\mathcal{C}}$ [111, 47] we know that a $\hat{\mathcal{C}}$-multiplet cannot appear in the OPE of $\hat{\mathcal{C}}\times\hat{\mathcal{B}}$. It seems puzzling that we obtain such a solution. But it is already known that this solution, although satisfies (5.28-5.34), it is not symmetric under $z_1\leftrightarrow z_3$, which comes from the uniqueness of the stress-tensor. Thus, the proportionality constant of (5.55) has to be 0 (see section 3.3.3 of [111].) This is the only case where there is another condition besides the constraints that comes from the EOM of the $\mathcal{J}$ and $\mathcal{L}_{(ij)}$ multiplets that is not satisfied.

Bibliography

[1] N. Arkani-Hamed, F. Cachazo, and J. Kaplan, *What is the Simplest Quantum Field Theory?*, JHEP **09** (2010) 016, `arXiv:0808.1446 [hep-th]`.

[2] P. H. Ginsparg, *APPLIED CONFORMAL FIELD THEORY*, in *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena Les Houches, France, June 28-August 5, 1988*. 1988. `arXiv:hep-th/9108028 [hep-th]`. `https://inspirehep.net/record/265020/files/arXiv:hep-th_9108028.pdf`.

[3] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, Nucl. Phys. **B241** (1984) 333–380.

[4] S. Ferrara, A. F. Grillo, and R. Gatto, *Tensor representations of conformal algebra and conformally covariant operator product expansion*, Annals Phys. **76** (1973) 161–188.

[5] A. M. Polyakov, *Nonhamiltonian approach to conformal quantum field theory*, Zh. Eksp. Teor. Fiz. **66** (1974) 23–42.

[6] G. Mack, *Duality in quantum field theory*, Nucl. Phys. **B118** (1977) 445–457.

[7] D. Pappadopulo, S. Rychkov, J. Espin, and R. Rattazzi, *OPE Convergence in Conformal Field Theory*, Phys. Rev. **D86** (2012) 105043, `arXiv:1208.6449 [hep-th]`.

[8] S. Rychkov, *EPFL Lectures on Conformal Field Theory in D¿= 3 Dimensions*, arXiv:1601.05000 [hep-th].

[9] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, JHEP **0812** (2008) 031, arXiv:0807.0004 [hep-th].

[10] D. Simmons-Duffin, *TASI Lectures on the Conformal Bootstrap*, arXiv:1602.07982 [hep-th].

[11] J. Penedones, *TASI lectures on AdS/CFT*, arXiv:1608.04948 [hep-th].

[12] D. Poland, D. Simmons-Duffin, and A. Vichi, *Carving Out the Space of 4D CFTs*, JHEP **05** (2012) 110, arXiv:1109.5176 [hep-th].

[13] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *Solving the 3D Ising Model with the Conformal Bootstrap*, Phys. Rev. **D86** (2012) 025022, arXiv:1203.6064 [hep-th].

[14] F. Kos, D. Poland, and D. Simmons-Duffin, *Bootstrapping the $O(N)$ vector models*, JHEP **06** (2014) 091, arXiv:1307.6856 [hep-th].

[15] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents*, J. Stat. Phys. **157** (2014) 869, arXiv:1403.4545 [hep-th].

[16] F. Kos, D. Poland, and D. Simmons-Duffin, *Bootstrapping Mixed Correlators in the 3D Ising Model*, JHEP **11** (2014) 109, arXiv:1406.4858 [hep-th].

[17] D. Simmons-Duffin, *A Semidefinite Program Solver for the Conformal Bootstrap*, JHEP **06** (2015) 174, arXiv:1502.02033 [hep-th].

[18] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. **38** (1999) 1113–1133, `arXiv:hep-th/9711200 [hep-th]`. [Adv. Theor. Math. Phys.2,231(1998)].

[19] J. Scherk and J. H. Schwarz, *Dual Models for Nonhadrons*, Nucl. Phys. **B81** (1974) 118–144.

[20] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B443** (1995) 85–126, `arXiv:hep-th/9503124 [hep-th]`.

[21] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, Phys. Rev. Lett. **75** (1995) 4724–4727, `arXiv:hep-th/9510017 [hep-th]`.

[22] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, Nucl. Phys. **B72** (1974) 461.

[23] H. Nastase, *Introduction to the ADS/CFT Correspondence.* Cambridge University Press, Cambridge, 2015.

[24] K. Lee and W. Siegel, *Conquest of the ghost pyramid of the superstring*, JHEP **08** (2005) 102, `arXiv:hep-th/0506198 [hep-th]`.

[25] K. Lee and W. Siegel, *Simpler superstring scattering*, JHEP **06** (2006) 046, `arXiv:hep-th/0603218 [hep-th]`.

[26] N. Berkovits, *Super Poincare covariant quantization of the superstring*, JHEP **04** (2000) 018, `arXiv:hep-th/0001035 [hep-th]`.

[27] W. Siegel, *Classical Superstring Mechanics*, Nucl. Phys. **B263** (1986) 93–104.

[28] L. Mazzucato, *Superstrings in AdS*, Phys. Rept. **521** (2012) 1–68, `arXiv:1104.2604 [hep-th]`.

[29] C. R. Mafra, O. Schlotterer, and S. Stieberger, *Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation*, Nucl. Phys. **B873** (2013) 419–460, `arXiv:1106.2645 [hep-th]`.

[30] P. Dai, R.-N. Huang, and W. Siegel, *Covariant propagator in AdS(5) x S**5 superspace*, JHEP **03** (2010) 001, `arXiv:0911.2211 [hep-th]`.

[31] H. Dorn, M. Salizzoni, and C. Sieg, *On the propagator of a scalar field on AdS x S and on the BMN plane wave*, JHEP **02** (2005) 047, `arXiv:hep-th/0307229 [hep-th]`.

[32] N. Beisert *et al.*, *Review of AdS/CFT Integrability: An Overview*, Lett. Math. Phys. **99** (2012) 3–32, `arXiv:1012.3982 [hep-th]`.

[33] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP **10** (2008) 091, `arXiv:0806.1218 [hep-th]`.

[34] O. Aharony, A. Fayyazuddin, and J. M. Maldacena, *The Large N limit of N=2, N=1 field theories from three-branes in F theory*, JHEP **07** (1998) 013, `arXiv:hep-th/9806159 [hep-th]`.

[35] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, Nucl.Phys. **B426** (1994) 19–52, `arXiv:hep-th/9407087 [hep-th]`.

[36] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484–550, `arXiv:hep-th/9408099 [hep-th]`.

[37] P. C. Argyres and M. R. Douglas, *New phenomena in SU(3) supersymmetric gauge theory*, Nucl. Phys. **B448** (1995) 93–126, `arXiv:hep-th/9505062 [hep-th]`.

[38] P. C. Argyres, M. R. Plesser, N. Seiberg, and E. Witten, *New N=2 superconformal field theories in four-dimensions*, Nucl. Phys. **B461** (1996) 71–84, `arXiv:hep-th/9511154 [hep-th]`.

[39] D. Gaiotto, *N=2 dualities*, JHEP **08** (2012) 034, `arXiv:0904.2715 [hep-th]`.

[40] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing, Hitchin Systems, and the WKB Approximation*, `arXiv:0907.3987 [hep-th]`.

[41] L. Bhardwaj and Y. Tachikawa, *Classification of 4d N=2 gauge theories*, JHEP **12** (2013) 100, `arXiv:1309.5160 [hep-th]`.

[42] C. Beem, M. Lemos, P. Liendo, L. Rastelli, and B. C. van Rees, *The $\mathcal{N} = 2$ superconformal bootstrap*, `arXiv:1412.7541 [hep-th]`.

[43] I. Ramirez and B. C. Vallilo, *Worldsheet dilatation operator for the AdS superstring*, JHEP **05** (2016) 129, `arXiv:1509.00769 [hep-th]`.

[44] F. Wegner, *Anomalous dimensions of high-gradient operators in then-vector model in $2+\epsilon$ dimensions*, Zeitschrift für Physik B Condensed Matter **78** no. 1, 33–43. `http://dx.doi.org/10.1007/BF01317354`.

[45] I. Ramirez and B. C. Vallilo, *Supertwistor description of the AdS pure spinor string*, Phys. Rev. **D93** (2016) no. 8, 086008, `arXiv:1510.08823 [hep-th]`.

[46] R. Roiban and W. Siegel, *Superstrings on AdS(5) x S**5 supertwistor space*, JHEP **11** (2000) 024, `arXiv:hep-th/0010104 [hep-th]`.

[47] P. Liendo, I. Ramirez, and J. Seo, *Stress-tensor OPE in $\mathcal{N} = 2$ superconformal theories*, JHEP **02** (2016) 019, `arXiv:1509.00033 [hep-th]`.

[48] I. A. Ramírez, *Mixed OPEs in $\mathcal{N} = 2$ superconformal theories*, JHEP **05** (2016) 043, `arXiv:1602.07269 [hep-th]`.

[49] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar $N = 4$ Supersymmetric Yang-Mills Theory: TBA and excited states*, Lett. Math. Phys. **91** (2010) 265–287, `arXiv:0902.4458 [hep-th]`.

[50] N. Beisert, *The complete one loop dilatation operator of N=4 superYang-Mills theory*, Nucl. Phys. **B676** (2004) 3–42, `arXiv:hep-th/0307015 [hep-th]`.

[51] N. Beisert and M. Staudacher, *The N=4 SYM integrable super spin chain*, Nucl. Phys. **B670** (2003) 439–463, `arXiv:hep-th/0307042 [hep-th]`.

[52] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Quantum Spectral Curve for Planar $\mathcal{N} =$ Super-Yang-Mills Theory*, Phys. Rev. Lett. **112** (2014) no. 1, 011602, `arXiv:1305.1939 [hep-th]`.

[53] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Quantum spectral curve for arbitrary state/operator in AdS_5/CFT_4*, JHEP **09** (2015) 187, `arXiv:1405.4857 [hep-th]`.

[54] N. Gromov, F. Levkovich-Maslyuk, G. Sizov, and S. Valatka, *Quantum spectral curve at work: from small spin to strong coupling in $\mathcal{N} = 4$ SYM*, JHEP **07** (2014) 156, `arXiv:1402.0871 [hep-th]`.

[55] M. Alfimov, N. Gromov, and V. Kazakov, *QCD Pomeron from AdS/CFT Quantum Spectral Curve*, JHEP **07** (2015) 164, `arXiv:1408.2530 [hep-th]`.

[56] C. Marboe and D. Volin, *Quantum spectral curve as a tool for a perturbative quantum field theory*, Nucl. Phys. **B899** (2015) 810–847, `arXiv:1411.4758 [hep-th]`.

[57] C. Marboe, V. Velizhanin, and D. Volin, *Six-loop anomalous dimension of twist-two operators in planar $\mathcal{N} = 4$ SYM theory*, JHEP **07** (2015) 084, `arXiv:1412.4762 [hep-th]`.

[58] N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, *Quantum Spectral Curve and the Numerical Solution of the Spectral Problem in AdS5/CFT4*, `arXiv:1504.06640 [hep-th]`.

[59] N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, *Pomeron Eigenvalue at Three Loops in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory*, Phys. Rev. Lett. **115** (2015) no. 25, 251601, `arXiv:1507.04010 [hep-th]`.

[60] I. Bena, J. Polchinski, and R. Roiban, *Hidden symmetries of the AdS(5) x S**5 superstring*, Phys. Rev. **D69** (2004) 046002, `arXiv:hep-th/0305116 [hep-th]`.

[61] B. C. Vallilo, *Flat currents in the classical AdS(5) x S**5 pure spinor superstring*, JHEP **03** (2004) 037, `arXiv:hep-th/0307018 [hep-th]`.

[62] N. Berkovits, *Quantum consistency of the superstring in AdS(5) x S**5 background*, JHEP **03** (2005) 041, `arXiv:hep-th/0411170 [hep-th]`.

[63] R. Benichou, *The Hirota equation for string theory in AdS5xS5 from the fusion of line operators*, Fortsch. Phys. **60** (2012) 896–900, `arXiv:1202.0084 [hep-th]`.

[64] Y. Aisaka, L. I. Bevilaqua, and B. C. Vallilo, *On semiclassical analysis of pure spinor superstring in an AdS_5 x S^5 background*, JHEP **09** (2012) 068, `arXiv:1206.5134 [hep-th]`.

[65] A. Cagnazzo, D. Sorokin, A. A. Tseytlin, and L. Wulff, *Semiclassical equivalence of Green-Schwarz and Pure-Spinor/Hybrid formulations of superstrings in AdS(5) x S(5) and AdS(2) x S(2) x T(6)*, J. Phys. **A46** (2013) 065401, `arXiv:1211.1554 [hep-th]`.

[66] M. Tonin, *On Semiclassical Equivalence of Green-Schwarz and Pure Spinor Strings in AdS(5) x S(5)*, J. Phys. **A46** (2013) 245401, `arXiv:1302.2488 [hep-th]`.

[67] B. C. Vallilo and L. Mazzucato, *The Konishi multiplet at strong coupling*, JHEP **12** (2011) 029, `arXiv:1102.1219 [hep-th]`.

[68] M. Heinze, *Spectrum and Quantum Symmetries of the* $\mathrm{AdS}_5 \times \mathrm{S}^5$ *Superstring*. PhD thesis, Humboldt U., Berlin, 2014. arXiv:1507.03005 [hep-th]. http://inspirehep.net/record/1382365/files/arXiv:1507.03005.pdf.

[69] A. Mikhailov, *Symmetries of massless vertex operators in AdS(5) x S**5*, J. Geom. Phys. **62** (2012) 479–505, arXiv:0903.5022 [hep-th].

[70] A. Mikhailov, *Finite dimensional vertex*, JHEP **12** (2011) 005, arXiv:1105.2231 [hep-th].

[71] N. Berkovits and T. Fleury, *Harmonic Superspace from the* $AdS_5 \times S^5$ *Pure Spinor Formalism*, JHEP **03** (2013) 022, arXiv:1212.3296 [hep-th].

[72] O. Chandia, A. Mikhailov, and B. C. Vallilo, *A construction of integrated vertex operator in the pure spinor sigma-model in* $AdS_5 \times S^5$, JHEP **11** (2013) 124, arXiv:1306.0145 [hep-th].

[73] N. Berkovits and O. Chandia, *Massive superstring vertex operator in D = 10 superspace*, JHEP **08** (2002) 040, arXiv:hep-th/0204121 [hep-th].

[74] C. Candu, V. Mitev, and V. Schomerus, *Spectra of Coset Sigma Models*, Nucl. Phys. **B877** (2013) 900–935, arXiv:1308.5981 [hep-th].

[75] A. Cagnazzo, V. Schomerus, and V. Tlapak, *On the Spectrum of Superspheres*, JHEP **03** (2015) 013, arXiv:1408.6838 [hep-th].

[76] A. A. Tseytlin, *On semiclassical approximation and spinning string vertex operators in AdS(5) x S**5*, Nucl. Phys. **B664** (2003) 247–275, arXiv:hep-th/0304139 [hep-th].

[77] R. R. Metsaev and A. A. Tseytlin, *Type IIB superstring action in AdS(5) x S**5 background*, Nucl. Phys. **B533** (1998) 109–126, arXiv:hep-th/9805028 [hep-th].

[78] L. Mazzucato and B. C. Vallilo, *On the Non-renormalization of the AdS Radius*, JHEP **09** (2009) 056, `arXiv:0906.4572 [hep-th]`.

[79] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, JHEP **09** (2004) 047, `arXiv:hep-th/0406055 [hep-th]`.

[80] N. Berkovits, *Simplifying and Extending the AdS(5) x S**5 Pure Spinor Formalism*, JHEP **09** (2009) 051, `arXiv:0812.5074 [hep-th]`.

[81] N. Berkovits and O. Chandia, *Superstring vertex operators in an AdS(5) x S**5 background*, Nucl. Phys. **B596** (2001) 185–196, `arXiv:hep-th/0009168 [hep-th]`.

[82] O. A. Bedoya, L. I. Bevilaqua, A. Mikhailov, and V. O. Rivelles, *Notes on beta-deformations of the pure spinor superstring in AdS(5) x S(5)*, Nucl. Phys. **B848** (2011) 155–215, `arXiv:1005.0049 [hep-th]`.

[83] W. Siegel, *AdS/CFT in superspace*, `arXiv:1005.2317 [hep-th]`.

[84] O. Chandia, *A Note on the classical BRST symmetry of the pure spinor string in a curved background*, JHEP **07** (2006) 019, `arXiv:hep-th/0604115 [hep-th]`.

[85] O. Chandia and B. C. Vallilo, *Non-minimal fields of the pure spinor string in general curved backgrounds*, JHEP **02** (2015) 092, `arXiv:1412.1030 [hep-th]`.

[86] J. H. Schwarz, *New formulation of the type IIB superstring action in* $AdS_5\times S^5$, JHEP **09** (2015) 041, `arXiv:1506.07706 [hep-th]`.

[87] W. Siegel, *Parametrization of cosets for AdS5xS5 superstring action*, `arXiv:1506.08172 [hep-th]`.

[88] N. Berkovits and P. S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, Nucl. Phys. **B635** (2002) 75–105, `arXiv:hep-th/0112160 [hep-th]`.

[89] O. Chacaltana and J. Distler, *Tinkertoys for Gaiotto Duality*, JHEP **11** (2010) 099, arXiv:1008.5203 [hep-th].

[90] P. Argyres, M. Lotito, Y. Lü, and M. Martone, *Geometric constraints on the space of N=2 SCFTs I: physical constraints on relevant deformations*, arXiv:1505.04814 [hep-th].

[91] D. Xie and S.-T. Yau, *4d N=2 SCFT and singularity theory Part I: Classification*, arXiv:1510.01324 [hep-th].

[92] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, and B. C. van Rees, *Infinite Chiral Symmetry in Four Dimensions*, Commun. Math. Phys. **336** (2015) no. 3, 1359–1433, arXiv:1312.5344 [hep-th].

[93] F. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, Nucl.Phys. **B599** (2001) 459–496, arXiv:hep-th/0011040 [hep-th].

[94] F. Dolan and H. Osborn, *Conformal partial waves and the operator product expansion*, Nucl.Phys. **B678** (2004) 491–507, arXiv:hep-th/0309180 [hep-th].

[95] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, *Spinning Conformal Correlators*, JHEP **1111** (2011) 071, arXiv:1107.3554 [hep-th].

[96] F. Dolan and H. Osborn, *Conformal Partial Waves: Further Mathematical Results*, arXiv:1108.6194 [hep-th].

[97] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, *Spinning Conformal Blocks*, JHEP **1111** (2011) 154, arXiv:1109.6321 [hep-th].

[98] D. Simmons-Duffin, *Projectors, Shadows, and Conformal Blocks*, JHEP **1404** (2014) 146, arXiv:1204.3894 [hep-th].

[99] A. Dymarsky, *On the four-point function of the stress-energy tensors in a CFT*, arXiv:1311.4546 [hep-th].

[100] M. Hogervorst, H. Osborn, and S. Rychkov, *Diagonal Limit for Conformal Blocks in d Dimensions*, JHEP **1308** (2013) 014, arXiv:1305.1321.

[101] A. L. Fitzpatrick, J. Kaplan, Z. U. Khandker, D. Li, D. Poland, and D. Simmons-Duffin, *Covariant Approaches to Superconformal Blocks*, JHEP **08** (2014) 129, arXiv:1402.1167 [hep-th].

[102] Z. U. Khandker, D. Li, D. Poland, and D. Simmons-Duffin, *$\mathcal{N}$ = 1 superconformal blocks for general scalar operators*, JHEP **08** (2014) 049, arXiv:1404.5300 [hep-th].

[103] E. Elkhidir, D. Karateev, and M. Serone, *General Three-Point Functions in 4D CFT*, arXiv:1412.1796 [hep-th].

[104] M. S. Costa and T. Hansen, *Conformal correlators of mixed-symmetry tensors*, arXiv:1411.7351 [hep-th].

[105] A. C. Echeverri, E. Elkhidir, D. Karateev, and M. Serone, *Deconstructing Conformal Blocks in 4D CFT*, arXiv:1505.03750 [hep-th].

[106] A. Bissi and T. Lukowski, *Revisiting N=4 superconformal blocks*, arXiv:1508.02391 [hep-th].

[107] R. Doobary and P. Heslop, *Superconformal partial waves in Grassmannian field theories*, arXiv:1508.03611 [hep-th].

[108] F. Rejon-Barrera and D. Robbins, *Scalar-Vector Bootstrap*, arXiv:1508.02676 [hep-th].

[109] F. A. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, Annals Phys. **307** (2003) 41–89, arXiv:hep-th/0209056 [hep-th].

[110] D. M. Hofman and J. Maldacena, *Conformal collider physics: Energy and charge correlations*, JHEP **05** (2008) 012, arXiv:0803.1467 [hep-th].

[111] S. M. Kuzenko and S. Theisen, *Correlation functions of conserved currents in N=2 superconformal theory*, Class. Quant. Grav. **17** (2000) 665–696, arXiv:hep-th/9907107 [hep-th].

[112] H. Osborn, *N=1 superconformal symmetry in four-dimensional quantum field theory*, Annals Phys. **272** (1999) 243–294, arXiv:hep-th/9808041 [hep-th].

[113] J.-H. Park, *Superconformal symmetry and correlation functions*, Nucl. Phys. **B559** (1999) 455–501, arXiv:hep-th/9903230 [hep-th].

[114] G. Arutyunov, B. Eden, and E. Sokatchev, *On nonrenormalization and OPE in superconformal field theories*, Nucl. Phys. **B619** (2001) 359–372, arXiv:hep-th/0105254 [hep-th].

[115] F. A. Dolan and H. Osborn, *Superconformal symmetry, correlation functions and the operator product expansion*, Nucl. Phys. **B629** (2002) 3–73, arXiv:hep-th/0112251 [hep-th].

[116] M. Nirschl and H. Osborn, *Superconformal Ward identities and their solution*, Nucl. Phys. **B711** (2005) 409–479, arXiv:hep-th/0407060 [hep-th].

[117] F. A. Dolan, L. Gallot, and E. Sokatchev, *On four-point functions of 1/2-BPS operators in general dimensions*, JHEP **09** (2004) 056, arXiv:hep-th/0405180 [hep-th].

[118] D. Poland and D. Simmons-Duffin, *Bounds on 4D Conformal and Superconformal Field Theories*, JHEP **05** (2011) 017, arXiv:1009.2087 [hep-th].

[119] M. Berkooz, R. Yacoby, and A. Zait, *Bounds on* $\mathcal{N}=1$ *superconformal theories with global symmetries*, JHEP **08** (2014) 008, arXiv:1402.6068 [hep-th]. [Erratum: JHEP01,132(2015)].

[120] J.-F. Fortin, K. Intriligator, and A. Stergiou, *Current OPEs in Superconformal Theories*, JHEP **09** (2011) 071, `arXiv:1107.1721 [hep-th]`.

[121] M. Bianchi, F. A. Dolan, P. J. Heslop, and H. Osborn, *N=4 superconformal characters and partition functions*, Nucl. Phys. **B767** (2007) 163–226, `arXiv:hep-th/0609179 [hep-th]`.

[122] C. Beem, L. Rastelli, and B. C. van Rees, *$\mathcal{W}$ symmetry in six dimensions*, JHEP **05** (2015) 017, `arXiv:1404.1079 [hep-th]`.

[123] C. Beem, W. Peelaers, L. Rastelli, and B. C. van Rees, *Chiral algebras of class S*, JHEP **05** (2015) 020, `arXiv:1408.6522 [hep-th]`.

[124] S. M. Chester, J. Lee, S. S. Pufu, and R. Yacoby, *Exact Correlators of BPS Operators from the 3d Superconformal Bootstrap*, JHEP **03** (2015) 130, `arXiv:1412.0334 [hep-th]`.

[125] M. Lemos and W. Peelaers, *Chiral Algebras for Trinion Theories*, JHEP **02** (2015) 113, `arXiv:1411.3252 [hep-th]`.

[126] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *Gauge Theories and Macdonald Polynomials*, Commun. Math. Phys. **319** (2013) 147–193, `arXiv:1110.3740 [hep-th]`.

[127] J. Maldacena and A. Zhiboedov, *Constraining Conformal Field Theories with A Higher Spin Symmetry*, J. Phys. **A46** (2013) 214011, `arXiv:1112.1016 [hep-th]`.

[128] V. Alba and K. Diab, *Constraining conformal field theories with a higher spin symmetry in d=4*, `arXiv:1307.8092 [hep-th]`.

[129] C. Beem, M. Lemos, L. Rastelli, and B. C. van Rees, *The $(2,0)$ superconformal bootstrap*, `arXiv:1507.05637 [hep-th]`.

[130] C. Beem, L. Rastelli, and B. C. van Rees, *The $\mathcal{N}=4$ Superconformal Bootstrap*, Phys. Rev. Lett. **111** (2013) 071601, arXiv:1304.1803 [hep-th].

[131] O. Aharony and Y. Tachikawa, *A Holographic computation of the central charges of d=4, N=2 SCFTs*, JHEP **01** (2008) 037, arXiv:0711.4532 [hep-th].

[132] L. Rastelli. *Private communication*.

[133] M. Buican and T. Nishinaka, *On the Superconformal Index of Argyres-Douglas Theories*, arXiv:1505.05884 [hep-th].

[134] C. Cordova and S.-H. Shao, *Schur Indices, BPS Particles, and Argyres-Douglas Theories*, arXiv:1506.00265 [hep-th].

[135] M. Lemos and P. Liendo, *Bootstrapping $\mathcal{N}=2$ chiral correlators*, JHEP **01** (2016) 025, arXiv:1510.03866 [hep-th].

[136] N. Bobev, S. El-Showk, D. Mazac, and M. F. Paulos, *Bootstrapping SCFTs with Four Supercharges*, arXiv:1503.02081 [hep-th].

[137] F. A. Dolan and H. Osborn, *Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to N=1 Dual Theories*, Nucl. Phys. **B818** (2009) 137–178, arXiv:0801.4947 [hep-th].

[138] M. Lemos and P. Liendo, *$\mathcal{N}=2$ central charge bounds from 2d chiral algebras*, JHEP **04** (2016) 004, arXiv:1511.07449 [hep-th].

[139] J. A. Minahan and D. Nemeschansky, *An N=2 superconformal fixed point with E(6) global symmetry*, Nucl. Phys. **B482** (1996) 142–152, arXiv:hep-th/9608047 [hep-th].

[140] J. A. Minahan and D. Nemeschansky, *Superconformal fixed points with E(n) global symmetry*, Nucl. Phys. **B489** (1997) 24–46, arXiv:hep-th/9610076 [hep-th].

[141] Y.-K. E. Cheung, O. J. Ganor, and M. Krogh, *Correlators of the global symmetry currents of 4-D and 6-D superconformal theories*, Nucl. Phys. **B523** (1998) 171–192, arXiv:hep-th/9710053 [hep-th].

[142] M. Buican, T. Nishinaka, and C. Papageorgakis, *Constraints on chiral operators in $\mathcal{N} = 2$ SCFTs*, JHEP **12** (2014) 095, arXiv:1407.2835 [hep-th].

[143] P. C. Argyres, M. Lotito, Y. Lü, and M. Martone, *Geometric constraints on the space of N=2 SCFTs II: Construction of special Kähler geometries and RG flows*, arXiv:1601.00011 [hep-th].

[144] M. F. Sohnius, *The Multiplet of Currents for $N = 2$ Extended Supersymmetry*, Phys. Lett. **B81** (1979) 8.

[145] J. Song, *Superconformal indices of generalized Argyres-Douglas theories from 2d TQFT*, arXiv:1509.06730 [hep-th].

[146] V. K. Dobrev and V. B. Petkova, *All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry*, Phys. Lett. **B162** (1985) 127–132.